Geometrical Construction for the Limiting Centres of a Cubic

BY

Gurudas Bhar.

In my paper on "The Osculating Conic at Infinity" (*Bulletin* *). M. S., Vol. XII, No. 4)*, I defined a *limiting centre* of a curve as the entre of a conic osculating the curve at infinity and proved the theorem hat *the three limiting centres of a cubic with three real asymptotes are ollinear.* The following simple geometrical construction for the limiting entres of the cubic will be found interesting It furnishes at the ame time an elegant proof of the above theorem.

Suppose the three real asymptotes BC, AB and AC meet the curve gain at P, Q and R respectively. Then P, Q, R are known to be ollinear. We shall show that the corresponding limiting centres ', Q' and R' which lie on BC, AB and AC respectively are such hat

$$PC = P'B$$

$$QB = Q'A$$

$$RC = R'A.$$

Let the equation of the cubic be as before

$$(y+\alpha_1 x-\beta_1)(y-\alpha_2 x)y+ly+mx+n=0;$$

t A be the origin; BC, AB and AC be the asymptotes

$$y=\alpha_1 x+\beta_1,$$

$$y=\alpha_2 x,$$

nd

$$y=0$$

espectively. If the co-ordinates of P, Q, R be (X_1, Y_1), (X_2, Y_2), (X_3, Y_3), those of P', Q', R' be (X_1', Y_1'), (X_2', Y_2'), (X_3', Y_3')

and those of A, B, C be (X_a, Y_a), (X_b, Y_b), (X_c, Y_c) we have

$$X_1 = -\frac{l\beta_1 + n}{l\alpha_1 + m},$$

$$Y_1 = \frac{m\beta_1 - n\alpha_1}{l\alpha_1 + m};$$

$$X_2 = -\frac{n}{l\alpha_2 + m},$$

$$Y_2 = -\frac{n\alpha_2}{l\alpha_2 + m};$$

$$X_3 = -\frac{n}{m},$$

$$Y_3 = 0.$$

Also (see my paper on *The Osculating Conic at Infinity*).

$$X_1' = \frac{-\alpha_1\ \beta_1\ (l\alpha_1 + m) + (\alpha_1 - \alpha_2)\ (n\alpha_1 - m\beta_1)}{\alpha_1\ (\alpha_1 - \alpha_2)\ (l\alpha_1 + m)}$$

$$Y_1' = \frac{-\alpha_2\ \beta_1\ (l\alpha_1 + m) + (\alpha_1 - \alpha_2)\ (n\alpha_1 - m\beta_1)}{(\alpha_1 - \alpha_2)\ (l\alpha_1 + m)}$$

$$X_2' = \frac{n\ (\alpha_1 - \alpha_2) - \beta_1\ (l\alpha_2 + m)}{(\alpha_1 - \alpha_2)\ (l\alpha_2 + m)},$$

$$Y_2' = \frac{n\alpha_2\ (\alpha_1 - \alpha_2) - \alpha_2\ \beta_1\ (l\alpha_2 + m)}{(\alpha_1 - \alpha_2)\ (l\alpha_2 + m)};$$

$$X_3' = \frac{n\alpha_1 - m\beta_1}{m\alpha_1},$$

$$Y_3' = 0.$$

Further

$$X_a = 0,$$

$$Y_a = 0;$$

$$X_b = -\frac{\beta_1}{\alpha_1 - \alpha_2},$$

$$Y_b = -\frac{\alpha_2\ \beta_1}{\alpha_1 - \alpha_2};$$

$$X_c = -\frac{\beta_1}{\alpha_1},$$

$$Y_c = 0.$$

Therefore

$$X_c-X_1=-\frac{\beta_1}{a_1}+\frac{l\beta_1+n}{la_1+m}=\frac{na_1-m\beta_1}{a_1\ (la_1+m)},$$

$$X_1'-X_b=\frac{-a_1\ \beta_1\ (la_1+m)+(a_1-a_2)\ (na_1-m\beta_1)}{a_1\ (a_1-a_2)\ (la_1+m)}+\frac{\beta_1}{a_1-a_2}$$

$$=\frac{na_1-m\beta_1}{a_1\ (la_1+m)},$$

or $\quad X_c-X_1=X_1'-X_b\,;$

and $\quad Y_c-Y_1=\frac{na_1-m\beta_1}{la_1+m},$

$$Y_1'-Y_b=\frac{-a_2\ \beta_1\ (la_1+m)+(a_1-a_2)\ (na_1-m\beta_1)}{(a_1-a_2)\ (la_1+m)}+\frac{a_2\ \beta_1}{a_1-a_2}$$

$$=\frac{na_1-m\beta_1}{la_1+m},$$

or $\quad Y_c-Y_1=Y_1'-Y_b\,;$

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hence $\quad$ PC=P′B.

Again $\quad X_b-X_2=-\frac{\beta_1}{a_1-a_2}+\frac{n}{la_2+m}$

$$=\frac{n(a_1-a_2)-\beta_1\ (la_2+m)}{(a_1-a_2)\ (la_2+m)},$$

$$X_2'-X_a=\frac{n(a_1-a_2)-\beta_1\ (la_2+m)}{(a_1-a_2)\ (la_2+m)},$$

or $\quad X_b-X_2=X_2'-X_a,$

and $\quad Y_b-Y_2=-\frac{a_2\ \beta_1}{a_1-a_2}+\frac{na_2}{la_2+m}$

$$=\frac{na_2\ (a_1-a_2)-a_2\ \beta_1\ (la_2+m)}{(a_1-a_2)\ (la_2+m)},$$

$$Y_2'-Y_a=\frac{na_2\ (a_1-a_2)-a_2\ \beta_1\ (la_2+m)}{(a_1-a_2)\ (la_2+m)},$$

or $\quad Y_b - Y_3 = Y_3{}' - Y_a$;

therefore $\quad QB = Q'A$.

Finally $\quad X_c - X_3 = -\dfrac{\beta_1}{\alpha_1} + \dfrac{n}{m} = \dfrac{n\alpha_1 - m\beta_1}{m\alpha_1}$,

$$X_3{}' - X_a = \frac{n\alpha_1 - m\beta_1}{m\alpha_1} \text{ ,}$$

or $\quad X_c - X_3 = X_3{}' - X_a$;

and $\quad Y_c - Y_3 = 0 = Y_3{}' - Y_a$;

therefore $\quad RC = R'A$.

It follows that the three limiting centres P′, Q′, R′ are collinear, as P, Q, R are collinear

In conclusion, my best thanks are due to Professor S. Mukhopadhyaya at whose suggestion and under whose guidance this paper was written.

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Note on Certain Properties of Legendre Polynominals of the Second Type

BY

K. Basu.

§. 1.

A study of the second type of solution of Legendre's differential equation has recently been made by Prof. Nicholson[1] who had to evoluate the definite integral $\int_{-1}^{1} [Q_n(\mu)]^2 \, d\mu$, in connection with the problem of two conducting disks: the capacity of an electrical condenser of this type depends on an integral involving Q functions. This short note is an attempt to investigate into some of its special features with reference to the analogous behaviour of the function $P_n(\mu)$. The function has been studied by Legendre,[2] Schläfli,[3] Heine,[4] Neumann,[5] Whittakar,[6] Nicholson[7] and others. Prof. Nicholson has established very recently certain important results of which I have made use of the following:—

$$(i)\quad \int_{-1}^{1} Q_{2m}(\mu)\, P_{2n+1}(\mu)\, d\mu = \frac{-2}{(2m-2n-1)\,(2m+2n+2)},$$

$$(ii)\quad \int_{-1}^{1} Q_{2m+1}(\mu)\, P_{2n}(\mu)\, d\mu = \frac{-2}{(2m-2n+1)\,(2m+2n+2)},$$

[1] Phil. Mag. Vol. 43. Jan. 1922.
[2] Calcul Integral, t. II.
[3] Über die zwei Heine'schen Kugelfunktionen (Bern, 1881).
[4] Crelle, Bd. XLII. Handbuch der Kugelfunktionen (Bern. 1878).
[5] Ibid, Bd. XXXVII, p. 24.
[6] Modern Analysis, Chap. XV, Second ed.
[7] Loc. cit.

$$(iii) \quad \int_0^1 Q_{2m}(\mu)\, P_{2n+1}(\mu)\, d\mu = \frac{-1}{(2m-2n-1)(2m+2n+2)},$$

$$(iv) \quad \int_0^1 Q_{2m+1}(\mu)\, P_{2n}(\mu)\, d\mu = \frac{-1}{(2m-2n+1)(2m+2n+2)}.$$

(v) $\displaystyle\int_{-1}^{1} P_p(\mu)\, Q_q(\mu)\, d\mu = 0$, in all cases in which p and q are both odd or both even.

$$(vi) \quad Q_{2m}(\mu) = -\sum_{n=0}^{\infty} \frac{(4n+3)\, P_{2n+1}(\mu)}{(2m-2n-1)(2m+2n+2)},$$

$$(vii) \quad Q_{2m+1}(\mu) = -\sum_{n=0}^{\infty} \frac{(4n+1)\, P_{2n}(\mu)}{(2m-2n+1)(2m+2n+2)},$$

the last two formulae are convergent when μ is between ± 1, both exclusive.

$$(viii) \quad Q^r_{2m}(\mu) = -\sum_{n=0}^{\infty} \frac{(4n+3)\, P^r_{2n+1}(\mu)}{(2m-2n-1)(2m+2n+2)},$$

$$(ix) \quad Q^r_{2m+1}(\mu) = -\sum_{n=0}^{\infty} \frac{(4n+1)\, P^r_{2n}(\mu)}{(2m-2n+1)(2m+2n+2)}.$$

§. 2.

When n is a positive integer or zero, $Q_n(\mu)$ is defined by :—

$$Q_n(\mu) = \frac{1}{2^{n+1}} \int_{-1}^{1} (1-t^2)^n (\mu-t)^{-n-1}\, dt.$$

this gives $\displaystyle\int_{-1}^{1} Q_n(\mu)\, d\mu = \frac{1}{2^{n+1}} \int_{-1}^{1}\int_{-1}^{1} (1-t^2)^n (\mu-t)^{-n-1}\, dt\, d\mu$

$$= -\frac{1+(-)^{n+1}}{n(n+1)}$$

$$= 0 \text{ or } -2/n(n+1),$$

according as n, is even or odd. This was obtained by Nicholson by another method. The recurrence formula

$$(n+1)\,Q_{n+1}(\mu)-(2n+1)\,\mu\,Q_n(\mu)+nQ_{n-1}(\mu)=0$$

gives $$\mu Q_{2m}(\mu)=\{(2m+1)\,Q_{2m+1}(\mu)+2mQ_{2m-1}(\mu)\}/4m+1.$$

or. $$\int_{-1}^{1}\mu Q_{2m}(\mu)\,d\mu=\frac{2m+1}{4m+1}\cdot\frac{-2}{(2m+1)\,(2m+2)}$$

$$+\frac{2m}{4m+1}\cdot\frac{-2}{(2m-1)\,2m}$$

$$=\frac{-2}{4m+1}\left\{\frac{1}{2m+2}+\frac{1}{2m-1}\right\}$$

$$=-1/(m-1)\,(2m-1).$$

also $$\int_{-1}^{1}\mu Q_{2m+1}(\mu)\,d\mu=\frac{2m+2}{4m+3}\cdot\int_{-1}^{1}Q_{2m+2}(\mu)\,d\mu$$

$$+\frac{2m+1}{4m+3}\int_{-1}^{1}Q_{2m}(\mu)\,d\mu=0.$$

that is $$\int_{-1}^{1}\mu Q_n(\mu)\,d\mu=\begin{cases}0, & \text{when } n \text{ is odd}\\ -2/(n+2)\,(n-1), & \text{when } n \text{ is even.}\end{cases}$$

Again $$(n+1)\,\mu Q_{n+1}(\mu)-(2n+1)\,\mu^2 Q_n(\mu)+n\mu\,Q_{n-1}(\mu)=0$$

$$\therefore (n+1)\int_{-1}^{1}\mu Q_{n+1}(\mu)-(2n+1)\int_{-1}^{1}\mu^2 Q_n(\mu)\,d\mu$$

$$+n\int\mu Q_{n-1}(\mu)\,d\mu=0$$

Suppose $n=2m$,

$$\therefore (2m+1)\int_{-1}^{1}\mu Q_{2m+1}(\mu)-(4m+1)\int_{-1}^{1}\mu^2 Q_{2m}(\mu)\,d\mu$$

$$+2m\int_{-1}^{1}\mu Q_{2m-1}(\mu)\,d\mu=0$$

$$\therefore \int_{-1}^{1}\mu^2 Q_{2m}(\mu)\,d\mu=0.$$

Next suppose $n=2m+1$,

$$\therefore (2m+2)\int_{-1}^{1}\mu Q_{2m+2}(\mu)\,d\mu-(4m+3)\int_{-1}^{1}\mu^2 Q_{2m+1}(\mu)\,d\mu$$

$$+(2m+1)\int_{-1}^{1}\mu Q_{2m}(\mu)=0.$$

i.e. $$\int_{-1}^{1}\mu^2 Q_{2m+1}(\mu)\,d\mu=\frac{(2m+2)}{4m+3}\cdot\frac{-2}{(2m+4)(2m+1)}$$

$$+\frac{(2m+1)}{4m+3}\cdot\frac{-2}{(2m+2)(2m-1)}$$

$$=-\frac{1}{4m+3}\cdot\left\{\frac{2m+2}{(m+2)(2m+1)}+\frac{2m+1}{(m+1)(2m-1)}\right\}.$$

Secondly take the recurrence formula :—

$$(\mu^2-1)\,Q'_n(\mu)=n\mu Q_n(\mu)-nQ_{n-1}(\mu).$$

Suppose $n=2m$

$$\text{thus} \int_{-1}^{1} (\mu^2-1)\, Q'_{2m}(\mu)\, d\mu = 2m \int_{-1}^{1} \mu Q_{2m}(\mu)\, d\mu - 2m \int_{-1}^{1} Q_{2m-1}(\mu)\, d\mu$$

$$= 2m \cdot \frac{-2}{(2m+2)(2m-1)} - 2m \cdot \frac{-2}{(2m-1)\,2m}$$

$$= \frac{-2m}{2m-1} \left\{ \frac{1}{(m+1)} - \frac{1}{m} \right\}$$

$$= \frac{-2m}{m(m+1)(2m-1)} = 2/(m+1)(2m-1)$$

Next, suppose $n=2m+1$,

$$\text{thus} \int_{-1}^{1} (\mu^2-1)\, Q'_{2m+1}(\mu)\, d\mu = (2m+1) \int_{-1}^{1} \mu Q_{2m+1}(\mu)\, d\mu - (2m+1)$$

$$\times \int_{-1}^{1} Q_{2m}(\mu)\, d\mu = 0.$$

$$\text{that is} \int_{-1}^{1} (\mu^2-1)\, Q'_n(\mu)\, d\mu = \begin{cases} 0, & \text{when } n \text{ is odd.} \\ 4/(n+2)(n-1), & \text{when } n \text{ is even} \end{cases}$$

Thirdly the recurrence formula

$$Q'_{n+1}(\mu) - Q'_{n-1}(\mu) = (2n+1)\, Q_n(\mu)$$

gives,

$$Q'_{n-1}(\mu) - Q'_{n-3}(\mu) = (2n-3)\, Q_{n-2}(\mu)$$

$$Q'_{n-3}(\mu) - Q'_{n-5}(\mu) = (2n-7)\, Q_{n-4}(\mu)$$

$$Q'_{n-5}(\mu) - Q'_{n-7}(\mu) = (2n-11)\, Q_{n-6}(\mu)$$

etc. etc.

$$\therefore Q'_{n+1}(\mu) = (2n+1)\, Q_n(\mu) + Q'_{n-1}(\mu)$$

$$= (2n+1)\, Q_n(\mu) + (2n-3)\, Q_{n-2}(\mu) + Q'_{n-3}(\mu)$$

$$= (2n+1)\, Q_n(\mu) + (2n-3)\, Q_{n-2}(\mu)$$

$$+ (2n-7)\, Q_{n-4}(\mu) + Q'_{n-5}(\mu)$$

and so on, until the last term is attained.

When n is even

$$\therefore Q'_{n+1}(\mu)=(2n+1)\,Q_n(\mu)+(2n-3)\,Q_{n-2}(\mu)+\ldots+Q_1'(\mu)$$

When n is odd

$$Q'_{n+1}(\mu)=(2n+1)\,Q_n(\mu)+(2n-3)\,Q_{n-2}(\mu)+\cdots+Q_0'(\mu)$$

Now
$$Q_0(\mu)=\tfrac{1}{2}\log\frac{\mu+1}{\mu-1}$$

$$Q_1(\mu)=\tfrac{1}{2}\,\mu\log\frac{\mu+1}{\mu-1}-1$$

$$\therefore Q_0'(\mu)=\frac{1}{1-\mu^2}.$$

$$Q_1'(\mu)=\tfrac{1}{2}\log\frac{\mu+1}{\mu-1}+\frac{\mu}{1-\mu^2}.$$

Hence
$$Q'_{n+1}(\mu)=\begin{cases}(2n+1)\,Q_n(\mu)+(2n-3)\,Q_{n-2}(\mu)+\ldots\\ \qquad\qquad+\dfrac{1}{1-\mu^2}.\ (n\text{ odd})\\ (2n+1)\,Q_n(\mu)+(2n-3)\,Q_{n-2}(\mu)+\ldots\\ \qquad\qquad+\tfrac{1}{2}\log\dfrac{\mu+1}{\mu-1}+\dfrac{\mu}{1-\mu^2}.\ (n\text{ even})\end{cases}$$

§. 3.

$$Q_{2m}(\mu)\,Q_{2r}(\mu)=\sum_{n=0}^{\infty}\frac{(4n+3)\,P_{2n+1}(\mu)}{(2m-2n-1)\,(2m+2n+2)}.$$
$$\sum_{p=0}^{\infty}\frac{(4p+3)\,P_{2p+1}}{(2r-2p-1)\,(2r+2p+2)}.$$

$$\therefore\int_{-1}^{1}Q_{2m}(\mu)\,Q_{2r}(\mu)\,d\mu$$

$$=\sum_{n=0}^{\infty}\frac{(4n+3)^2}{(2m-2n-1)\,(2m+2n+2)\,(2r-2p-1)\,(2r+2p+2)}.$$
$$\int_{-1}^{1}P^2_{2n+1}(\mu)\,d\mu$$

$$=2\sum_{n=0}^{\infty}\frac{4n+3}{(2m-2n-1)\,(2m+2n+2)\,(2r-2p-1)\,(2r+2p+2)}.$$

Similarly $\int_{-1}^{1} Q_{2m+1}(\mu)\, Q_{2r+1}(\mu)\, d\mu$

$$=2\sum_{n=0}^{\infty} \frac{4n+1}{(2m-2n+1)(2m+2n+2)(2r-2n+1)(2r+2n+2)}$$

$$\int_{-1}^{1} Q_{2m}(\mu)\, Q_{2r+1}(\mu)\, d\mu = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{4n+3}{(2m-2n-1)(2m+2n+2)}$$

$$\times \frac{4p+1}{(2r-2p+1)(2r+2p+2)} \int_{-1}^{1} P_{2n+1}(\mu)\, P_{2p}(\mu)\, d\mu$$

$$=0 \qquad \ldots \qquad \ldots \qquad \ldots \quad \text{(A)}$$

Again from
$$\int_{0}^{1} P_m(\mu)\, P_n(\mu)\, d\mu = \begin{cases} 1/2n+1 & (m=n) \\ 0 & (m-n \text{ even}) \\ (-)^{\mu+v} \dfrac{1}{2^{m+n-1}(n-m)(n+m+1)} \cdot \dfrac{n!\, m!}{(v!)^2\, (\mu!)^2} & \left(\begin{matrix} n=2v+1 \\ \text{or} \\ m=2\mu \end{matrix}\right) \end{cases}$$

$\int_{0}^{1} Q_{2m}(\mu)\, Q_{2r}(\mu)\, d\mu$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(4n+3)(4p+3)}{(2m-2n-1)(2m+2n+2)(2r-2p-1)(2r+2p+2)} \times$$

$$\int_{0}^{1} P_{2n+1}(\mu)\, P_{2p+1}(\mu)\, d\mu = 0$$

Similarly, $\int_{0}^{1} Q_{2m+1}(\mu)\, Q_{2r+1}(\mu) = 0$

$$\int_0^1 Q_{2m}(\mu)\, Q_{2r+1}(\mu)\, d\mu$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(4n+3)\,(4p+1)}{(2m-2n-1)\,(2m+2n+2)\,(2r-2p+1)\,(2r+2p+2)}$$

$$\times\int_0^1 P_{2n+1}(\mu)\, P_{2p}(\mu)\, d\mu$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(4n+3)\,(4p+1)}{(2m-2n-1)\,(2m+2n+2)\,(2r-2p+1)\,(2r+2p+2)}$$

$$\times\frac{(-)^{n+p}}{2^{2(n+p)}\,(2n-2p+1)\,(2n+2p+2)}\,\frac{(2p)!\,(2n+1)!}{(n!)^2\,(p!)^2}$$

Similar result for $\int_0^1 Q_{2m+1}(\mu)\, Q_{2r}(\mu)\, d\mu$.

Lastly, since

$$\int_{-1}^{1} P_n^m(\mu)\, P_r^m(\mu)\, d\mu=\begin{cases}0 & r\neq n\\[4pt] \dfrac{2}{2n+1}\,\dfrac{(n+m)!}{(n-m)!} & r=n.\end{cases}$$

$$\int_{-1}^{1} Q^r_{2m}(\mu)\, Q^r_{2s}(\mu)\, d\mu$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(4n+3)\,(4p+3)}{(2m-2n-1)\,(2m+2n+2)\,(2s-2p-1)\,(2s+2p+2)}$$

$$\times\int_{-1}^{1} P^r_{2n+1}(\mu)\, P^r_{2p+1}(\mu)\, d\mu$$

$$=\sum_{n=0}^{\infty}\frac{(4n+3)^2}{(2m-2n-1)\,(2m+2n+2)\,(2s-2n-1)\,(2s+2n+2)}$$

$$\times\frac{2}{4n+3}\cdot\frac{(2n+1+r)!}{(2n-r+1)!}\text{, (r being an integer).}$$

$$=2\sum_{n=0}^{\infty}\frac{(4n+3)}{(2m-2n-1)\,(2m+2n+2)\,(2s-2n-1)\,(2s+2n+2)}$$

$$\times\frac{(2n+r+1)!}{(2n-r+1)!}$$

Similarly. $\displaystyle\int_{-1}^{1} Q^r_{2m+1}(\mu)\, Q^r_{2s+1}(\mu)\, d\mu$

$$=\sum_{n=0}^{\infty}\frac{(4n+1)^2}{(2m-2n+1)\,(2m+2n+2)\,(2s-2n+1)\,(2s+2n+2)}$$

$$\times\frac{2}{4n+1}\cdot\frac{(2n+r)!}{(2n-r)!}$$

$$=\sum_{n=0}^{\infty}\frac{(4n+1)}{(2m-2n+1)(2m+2n+2)(2s-2n+1)(2s+2n+2)}\cdot\frac{(2n+r)!}{(2n-r)!}$$

It can be proved very easily from above

$$\int_{-1}^{1} Q^r_{2m}(\mu)\, Q^r_{2s-1}(\mu)\, d\mu=0$$

$$\int_{-1}^{1} Q^r_{2m+1}(\mu)\, Q^r_{2s}(\mu)\, d\mu=0.$$

$$\int_{-1}^{1}[Q^r_{2n}(\mu)]^2\,d\mu$$

$$=2\sum_{n=0}^{\infty}\frac{4n+3}{(2m-2n-1)^2\,(2m+2n+2)^2}\;\frac{(2n+r+1)!}{(2n-r+1)!}$$

$$\int_{-1}^{1}[Q^r_{2n+1}(\mu)]^2\,d\mu$$

$$=2\sum_{n=0}^{\infty}\frac{4n+1}{(2m-2n+1)^2\,(2m+2n+2)^2}\cdot\frac{(2n+r)!}{(2n-r)!}$$

§. 4.

If m and n are positive integers and $m \leq n$, Adam[1] found out

$$P_m(z)\,P_n(z)=\sum_{r=0}^{\infty}\frac{A_{m-r}\,A_r\,A_{n-r}}{A_{n+m-r}}\left(\frac{2n+2m-4r+1}{2n+2m-2r+1}\right)P_{n+m-2r},$$

where $A_m=1\cdot3\cdot5\;\ldots\ldots\;(2m-1)/m!$

The above can be written in the form

$P_m(z)\,P_n(z)$

$$=\sum_{q=n-m}^{n+m}\frac{2q+1}{m+n+q+1}\cdot\frac{A_{\frac{n+m-q}{2}}\;A_{\frac{n+q-m}{2}}\;A_{\frac{m+q-n}{2}}}{A_{\frac{n+m+q}{2}}}\;P_q(z),$$

$$=\sum_{q=n-m}^{n+m}\cdot[A]_{n,m,q}\;P_q(z),$$

where $[A]_{n,m,q}$

$$=\frac{2q+1}{m+n+q+1}\cdot\frac{A_{\frac{n+m-q}{2}}\;A_{\frac{n+q-m}{2}}\;A_{\frac{m+q-n}{2}}}{A_{\frac{n+m+q}{2}}}.$$

[1] Proc. Roy. Soc. XXVII.

whence. $Q_{2m}(\mu)\, Q_{2s}(\mu)\, P_{2r}(\mu)$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(4n+3)\,(4p+3)}{(2m-2n-1)\,(2m+2n+2)\,(2s-2p-1)\,(2s+2p+2)}$$

$$\times P_{2n+1}(\mu)\, P_{2p+1}(\mu)\, P_{2r}(\mu)$$

Now $P_{2n+1}(\mu)\, P_{2p+1}(\mu)\, P_{2r}(\mu)$

$$=\sum_{q=n-m}^{n+m}[A]_{2n+1,\,2p+1,\,q}\cdot P_q(\mu)\, P_{2r}(\mu)$$

$$=\sum_{q=n-m}^{n+m}[A]_{2n+1,\,2p+1,\,q}\sum_{t=2r-q}^{2r+q}[A]_{2r,\,q,\,t}\, P_t(\mu),$$

if $r \geq n+p+1$

This can be expanded by the scheme pointed out by S. K. Banerji.[1]

Also $\int_{-1}^{1} Q_{2m}(\mu)\, Q_{2s}(\mu)\, P_{2r}(\mu)\, d\mu$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(4n+3)\,(4p+3)}{(2m-2n-1)\,(2m+2n+2)\,(2s-2p-1)\,(2s+2p+2)}$$

$$\times\int_{-1}^{1} P_{2n+1}(\mu)\, P_{2p+1}(\mu)\, P_{2r}(\mu)\, d\mu$$

where $\int_{-1}^{1} P_{2n+1}(\mu)\, P_{2p+1}(\mu)\, P_{2r}(\mu)$

$$=\frac{2}{2n+2p+2r+3}\,\frac{A_{\frac{2n+2p-2r+2}{2}}\, A_{\frac{2p+2r-2n}{2}}\, A_{\frac{2r-2n-2p-2}{2}}}{A_{\frac{2n+2p+2r+2}{2}}}$$

$$=\frac{2}{2n+2p+2r+3}\,\frac{A_{n+p-r+1}\, A_{p+r-n}\, A_{r-n-p-1}}{A_{n+p+r+1}}$$

[1] Bull. Cal. Math. Soc. Vol. XI, No. 3, 1920, p. 180.

[2] Banerji, *loc. cit.*

Similarly the values of $\int_{-1}^{1} Q_{2m+1}(\mu)\, Q_{2s+1}(\mu)\, P_{2r+1}(\mu)\, d\mu$,

$$\int_{-1}^{1} Q_{2m+1}(\mu)\, Q_{2s+1}(\mu)\, P_{2r}(\mu), \quad \int_{-1}^{1} Q_{2m}(\mu)\, Q_{2s+1}(\mu)\, P_{2r}(\mu), \text{ etc,}$$

can be written down.

Proceeding as before the values of

$$\int_{-1}^{1} Q_m(\mu)\, Q_n(\mu)\, Q_r(\mu), \qquad [m \neq n \neq r]$$

for all possible combinations of m, n, r, according as they are even or odd, can be found out in the form of series of functions of m, n, and r.

ON THE STEADY MOTION OF A VISCOUS FLUID DUE TO THE ROTATION OF TWO SPHEROIDS ABOUT THEIR COMMON AXIS OF REVOLUTION

BY

NRIPENDRANATH SEN, M.SC.

1. In a memoir,[1] Dr. G. B. Jeffery has completely solved several cases of steady motion of a viscous fluid due to various rotating bodies including the case of two spheres rotating about their line of centres. The object of the present paper is to present the solution of a more difficult problem *viz.* the problem of the steady motion of a viscous fluid due to the rotation of two spheroids, both prolate or both oblate, about their common axis of revolution. The problem has been completely solved first for two rotating prolate spheroids with no limitation as regards their eccentricities and central distance and the solution for the case of two rotating oblate spheroids has been deduced therefrom by suitable substitutions. The success of the problem depends on a transformation theorem in Spheroidal Harmonics[2] proved in (8), (9), (10) and (11) of the present paper.

The present problem in its much simpler aspect has been attempted in a previous issue of this bulletin.[3] The results obtained there are deducible as particular cases of the general problem here studied but they are found to differ widely in many respects from the results deduced from those obtained by me. The reason for this difference is due to mistakes in approximation in that paper on its author's part, which have been clearly pointed out and explained at length in (5) and (6) of the present paper, where the correct solutions of those problems have also been given.

[1] G. B. Jeffery.—"On the steady motion of a solid of revolution in a viscous fluid"—Proc. London Math. Soc. February, 1915.

[2] Also see Bibhutibhushan Datta.—"On a transformation theorem relating to spheroidal Harmonics"—Tôhuku. Math. Jour. Vol. 15, 166-171, 1919.

[3] Bijon Dutt.—"Bul. Cal. Math. Soc." Vol. 10, 43-53, 1918-19.

TWO PROLATE SPHEROIDS ROTATING ABOUT THEIR COMMON AXIS OF REVOLUTION.

2. Let $x_1=k_1\ (1-\mu_1^2)^{\frac{1}{2}}\ (\lambda_1^2-1)^{\frac{1}{2}} \cos w$;

$$y_1=k_1\ (1-\mu_1^2)^{\frac{1}{2}}\ (\lambda_1^2-1)^{\frac{1}{2}} \sin w;$$

$$z_1=k_1\ \mu_1\ \lambda_1 \qquad ;$$

$$x_2=k_2\ (1-\mu_2^2)^{\frac{1}{2}}\ (\lambda_2^2-1)^{\frac{1}{2}} \cos w.$$

$$y_2=k_2\ (1-\mu_2^2)^{\frac{1}{2}}\ (\lambda_2^2-1)^{\frac{1}{2}} \sin w.$$

$$z_2=k_2\ \mu_2\ \lambda_2$$

be the two systems of co-ordinates referred to the centres of the two spheroids as origin and (λ_1,μ_1,w) (λ_2,μ_2,w) be the two systems of ovary spheroidal co-ordinates so that $\lambda_1=\lambda_{10}$, $\lambda_2=\lambda_{20}$ on the surfaces of the given spheroids whose semi axes are a_1, c_1 $(a_1 > c_1)$ and a_2, c_2 $(a_2 > c_2)$ respectively. Also let s=distance between their centres so that $s > a_1+a_2$ and w_1, w_2 be their angular velocities of rotation.

If (ρ, Φ, z) be the cylindrical co-ordinates of a point, Dr. Jeffery[1] has shewn that v=velocity in the direction of ϕ,

$=f(\rho, z)$ where $v \sin\phi$ is a solution of Laplace's equation

$$\nabla^2\ (v \sin\phi)=0 \qquad \ldots \quad (1)$$

and

$$v=\rho_1\ w_1=k_1\ (1-\mu_1^2)^{\frac{1}{2}}\ (\lambda_1^2-1)^{\frac{1}{2}}\ w_1 \quad \text{over } \lambda_1=\lambda_{10} \qquad \ldots \quad (2)$$

$$v=\rho_2\ w_2=k_2\ (1-\mu_2^2)^{\frac{1}{2}}\ (\lambda_2^2-1)^{\frac{1}{2}}\ w_2 \quad \text{over } \lambda_2=\lambda_{20} \qquad \ldots \quad (3)$$

The problem is, therefore, to find an expression for v satisfying the conditions (1), (2), and (3) and the further condition $v=0$ at infinity

i.e. when $\lambda=\infty$ ... (4)

[1] Jeffery—"On the steady motion of a solid of revolution in a viscous fluid" Proc. Lond. Math. Soc. Feb. 1915.

Assume $v = \sum_{n=1}^{\infty} \{A_n P_n^1(\mu_1) Q_n^1(\lambda_1) + B_n P_n^1(\mu_2) Q_n^1(\lambda_2)\}$ (5)

where A_n, B_n are arbitrary constants.

Evidently (5) satisfies (1) and (4). To determine the sets of constants A_n, B_n so as to satisfy the boundary conditions (2) and (3).

Now, it may be proved that

$$P_n^{\sigma}(\mu_2) Q_n^{\sigma}(\lambda_2) =$$

$$(-)^{\sigma} \frac{(n+\sigma)!}{(n-\sigma)!} \sum_{m=\sigma}^{m=\infty} (2m+1) \frac{(m-\sigma)!}{(m+\sigma)!} \omega_1(m, n) P_m^{\sigma}(\mu_1) P_m^{\sigma}(\lambda_1) \qquad \ldots (6)$$

when $s > a_1$ and $> a_2$

$$\text{where} \quad \omega_1(m, n) = \tfrac{1}{2} \int_{-1}^{1} Q_n\left(\frac{t_1 - p}{\rho_1}\right) P_m(p)\, dp$$

$$= (-)^m \frac{2^m\, m!}{(2m+1)!} \rho_1^{m+1} \Big[D_{t_1}^m + \frac{\rho_1^2}{2\cdot(2m+3)} D_{t_1}^{m+2} + \ldots\Big] Q_n(t_1)$$

where $\rho_1^2 = \dfrac{a_2^2 - c_2^2}{a_1^2 - c_1^2} = \dfrac{a_2^2 e_2^2}{a_1^2 e_1^2} = \dfrac{k_2^2}{k_1^2}$ where e_1, e_2 are eccentricities of the generating ellipses of the two spheroids and

$$t_1 = \frac{s}{k_1} \text{ and } D_{t_1}^m = \frac{d^m}{d t_1^m} \qquad \ldots (7)$$

$$\text{For,} \int_{-\pi}^{\pi} Q_n\left(\frac{z_2 + i x_2 \cos u + i y_2 \sin u}{k_2}\right) \cos \sigma u\, du$$

$$= \int_{-\pi}^{\pi} Q_n\{\mu_2 \lambda_2 + (\mu_2^2 - 1)^{\frac{1}{2}} (\lambda_2^2 - 1)^{\frac{1}{2}} \cos(w-u)\} \cos \sigma u\, du$$

$$= 2\pi \frac{(n-\sigma)!}{(n+\sigma)!} P_n^{\sigma}(\mu_2) Q_n^{\sigma}(\lambda_2) \cos \sigma w \qquad \ldots (8)[1]$$

with Hobson's definition of associated Legendre's functions.

[1] Whittaker—'Mod. Analysis' 15·71, Example p. 323.

Choosing z-axes such that $z_2 = s - z_1$, we have

$$\frac{(z_2 + ix_2 \cos u + iy_2 \sin u)}{k_2} = \frac{k_1}{k_2} \cdot \frac{(s - z_1 + ix_1 \cos u + iy_1 \sin u)}{k_1}$$

Remembering that $\dfrac{1}{\left(\dfrac{z_2 + ix_2 \cos u + iy_2 \sin u}{k_2}\right)^{n+1}}$

$$= \rho_1^{n+1} \left(\frac{s}{k_1} - \frac{z_1 - ix_1 \cos u - iy_1 \sin u}{k_1} \right)^{-n-1}$$

$$= (-)^n \frac{\rho_1^{n+1}}{n!} \sum_{m=0}^{\infty} (2m+1) \mathrm{P}_m \left(\frac{z_1 - ix_1 \cos u - iy_1 \sin u}{k_1} \right) \frac{d^n}{dt_1^n} \mathrm{Q}_m (t_1) \qquad \dots (9)^1$$

defining ρ_1, t_1 as in (7), we have

$$\int_{-\pi}^{\pi} \mathrm{Q}_n \left(\frac{z_2 + ix_2 \cos u + iy_2 \sin u}{k_2} \right) \cos \sigma u \, du$$

$$= \int_{-\pi}^{\pi} (-)^n \frac{2^n \lfloor n}{\lfloor 2n+1} \rho_1^{n+1} \Bigg\{ \sum_{m=0}^{\infty} (2m+1) \mathrm{P}_m \left(\frac{z_1 - ix_1 \cos u - iy_1 \sin u}{k_1} \right) \frac{d^n}{dt_1^n} \mathrm{Q}_m (t_1),$$

$$+ \frac{\rho_1^2}{2 \cdot (2n+3)} \sum_{m=0}^{\infty} (2m+1) \mathrm{P}_m \left(\frac{z_1 - ix_1 \cos u - iy_1 \sin u}{k_1} \right) \frac{d^{n+2}}{dt_1^{n+2}} \mathrm{Q}_m (t_1)$$

$$+ \text{etc.} \Bigg\} \cos \sigma u \, du$$

$$= \sum_{m=0}^{\infty} \omega_1 (m, n) (2m+1) \mathrm{P}_m \left(\frac{z_1 - ix_1 \cos u - iy_1 \sin u}{k_1} \right) \cos \sigma u \, du$$

$$= 2\pi \sum_{m=\sigma}^{\infty} (-)^{\sigma} \frac{\lfloor m-\sigma}{\lfloor m+\sigma} (2m+1) \, \omega_1 (m, n) \, \mathrm{P}_m^{\sigma} (\mu_1) \, \mathrm{P}_n^{\sigma} (\lambda_1) \cos \sigma w \qquad \dots (10)$$

with Hobson's definition of associated Legendre Functions.

[1] Todhunter—"The functions of Laplace, Lamé and Bessel" p. 88.

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$$\therefore \int_{-\pi}^{\pi} P_n\left(\frac{z_1-ix_1\cos u-iy_1\sin u}{k_1}\right)\cos\sigma u\,du$$

$$=\int_{-\pi}^{\pi} P_n\{\mu_1\lambda_1-(\mu_1^2-1)^{\frac{1}{2}}(\lambda_1^2-1)^{\frac{1}{2}}\cos(w-u)\}\cos\sigma u\,du$$

$$=2\pi(-)^{\sigma}\frac{\lfloor m-\sigma}{\lfloor m+\sigma} P_m^{\sigma}(\mu_1)P_m^{\sigma}(\lambda_1)\cos\sigma w \quad \ldots \quad (11)^1$$

with Hobson's definition of the associated Legendre Functions.

$\therefore$ From (8) and (10), we have,

$$P_n^{\sigma}(\mu_2)Q_n^{\sigma}(\lambda_2)=\frac{\lfloor n+\sigma}{\lfloor n-\sigma}\sum_{m=\sigma}^{\infty}(-)^{\sigma}\frac{\lfloor m-\sigma}{\lfloor m+\sigma}(2m+1)\,\omega_1(m,n)\,P_m^{\sigma}(\mu_1)P_m^{\sigma}(\lambda_1)$$

with either Hobson's or Ferrer's definitions of associated Legendre Functions or with both the definitions simultaneously always taking care to stick to the definition or definitions used. Thus if $P_n^{\sigma}(\mu)$ be in Ferrer's definition and $Q_n^{\sigma}(\lambda)$ or $P_n^{\sigma}(\lambda)$ be in Hobson's definition,—which is usually the case the theorem (6) is still true.

Substituting the value of $P_n^1(\mu_2)Q_n^1(\lambda_2)$ in (5) from (6), we have

$$v=\sum_{n=1}^{\infty}\{A_n P_n^1(\mu_1)Q_n^1(\lambda_1)$$
$$-B_n\sum_{m=1}^{\infty}\frac{n(n+1)(2m+1)}{m(m+1)}\,\omega_1(m,n)\,P_m^1(\mu_1)P_m^1(\lambda_1)\}$$

Hence, from the surface condition (2),

$$v=k_1(\lambda_{10}^2-1)^{\frac{1}{2}}w_1(1-\mu_1^2)^{\frac{1}{2}}=k_1(\lambda_{10}^2-1)^{\frac{1}{2}}w_1P_1^1(\mu_1)$$
$$=\sum_{n=1}^{\infty}\{A_n P_n^1(\mu_1)Q_n^1(\lambda_{10})$$
$$-B_n\sum_{m=1}^{\infty}\frac{n(n+1)}{m(m+1)}(2m+1)\,\omega_1(m,n)\,P_m^1(\mu_1)P_m^1(\lambda_{10})\}$$

[1] Whittaker—Mod. Anal. 15·7.

This must be true at every point of the surface of the spheroid $\lambda_1 = \lambda_{10}$. Equating, therefore, the co-efficients of the various zonal harmonics of μ_1,

$$k_1 (\lambda_{10}^2 - 1)^{\frac{1}{2}} w_1 = A_1 Q_1^1 (\lambda_{10})$$

$$- \frac{3}{2} \sum_{n=1}^{\infty} n (n+1) \omega_1 (1, n) P_1^1 (\lambda_{10}) B_n \quad \ldots \quad (12)$$

$$0 = A_p Q_p^1 (\lambda_{10})$$

$$- \frac{2p+1}{p (p+1)} \sum_{n=1}^{\infty} n (n+1) \omega_1 (p, n) P_p^1 (\lambda_{10}) B_n \quad \ldots \quad (13)$$

$$(p = 2, 3, \ldots \text{ ad inf.}).$$

The corresponding set of equations giving B_1, B_2 etc. in terms of A's can be written down from symmetry from (12) and (13). Thus,

$$k_2 (\lambda_{20}^2 - 1)^{\frac{1}{2}} w_2 = B_1 Q_1^1 (\lambda_{20})$$

$$- \frac{3}{2} \sum_{n=1}^{\infty} n (n+1) \omega_2 (1, n) P_1^1 (\lambda_{20}) A_n \quad \ldots \quad (14)$$

$$0 = B_p Q_p^1 (\lambda_{20})$$

$$- \frac{2p+1}{p (p+1)} \sum_{n=1}^{\infty} n (n+1) \omega_2 (p, n) P_p^1 (\lambda_{20}) A_n \quad \ldots \quad (15)$$

$$(p = 2, 3, \ldots \text{ ad. inf.}).$$

$$\text{where } \omega_2 (m, n) = \frac{1}{2} \int_{-1}^{1} Q_n \left(\frac{t_2 - p}{\rho_2} \right) P_m (p) \, dp$$

$$= (-)^n \frac{2^n \lfloor n}{\lfloor 2n+1} \rho_2^{n+1} \left[D_{t_2}^n + \frac{\rho_2^2}{2 \cdot (2n+3)} D_{t_2}^{n+2} + \text{etc.} \right] Q_m (t_2)$$

$$\rho_2 = \frac{k_1}{k_2}, \; t_2 = \frac{s}{k_2}, \; D_{t_2}^n = \frac{d^n}{d_{t_2}^n} \quad \ldots \quad (16)$$

The two sets of equations (12); (13) and (14), (15) are, sufficient to determine the two sets of constants A_1, A_2, etc and B_1, B_2, etc., as will be shewn presently.

To determine A_1, A_2, etc., substitute the values of B_1, B_2, etc. in (12) and (13) from (14) and (15). After a little simplification,

$$A_1 - \sum_{n=1}^{\infty} \theta_{1n} A_n = d_1 \qquad \ldots \quad (17)$$

$$A_p - \sum_{n=1}^{\infty} \theta_{pn} A_n = dp \qquad (p=2, 3 \ldots \text{ad. inf.}) \qquad \ldots \quad (18)$$

where θ_{pn}

$$= \frac{n(n+1)(2p+1)}{p(p+1)} \frac{P_p^1(\lambda_{10})}{Q_p^1(\lambda_{10})} \sum_{m=1}^{\infty} (2m+1)\, \omega_1(p, m)\, \omega_2(m, n) \frac{P_m^1(\lambda_{20})}{Q_m^1(\lambda_{20})}$$

$$d_1 = k_1 w_1 \frac{P_1^1(\lambda_{10})}{Q_1^1(\lambda_{10})} + 3 k_2 w_2 \omega_1(1, 1) \frac{P_1^1(\lambda_{10})}{Q_1^1(\lambda_{10})} \frac{P_1^1(\lambda_{20})}{Q_1^1(\lambda_{20})}$$

$$d_p = \frac{2(2p+1)}{p(p+1)} k_2 w_2 \omega_1(p, 1) \frac{P_p^1(\lambda_{10})}{Q_p^1(\lambda_{10})} \frac{P_1^1(\lambda_{20})}{Q_1^1(\lambda_{20})}$$

$$(p=2, 3, \ldots \text{ad. inf.}) \qquad \ldots \quad (19)$$

The corresponding equations giving B_1, B_2, etc. can be written down easily from (17), (18) and (19) Thus, from symmetry,

$$B_1 - \sum_{n=1}^{\infty} \theta'_{1n} B_n = d'_1 \qquad \ldots \quad (20)$$

$$B_p - \sum_{n=1}^{\infty} \theta'_{pn} B_n = d'_p \qquad (p=2, 3, \ldots \text{ad inf.}) \qquad \ldots \quad (21)$$

where θ'_{pn}

$$= \frac{n(n+1)(2p+1)}{p(p+1)} \frac{P_p^1(\lambda_{20})}{Q_p^1(\lambda_{20})} \sum_{m=1}^{\infty} (2m+1)\, \omega_2(p, m)\, \omega_1(m, n) \frac{P_m^1(\lambda_{10})}{Q_m^1(\lambda_{10})}$$

$$d'_1 = k_2 w_2 \frac{P_1^1(\lambda_{20})}{Q_1^1(\lambda_{20})} + 3 k_1 w_1 \omega_2(1, 1) \frac{P_1^1(\lambda_{20})\, P_1^1(\lambda_{10})}{Q_1^1(\lambda_{20})\, Q_1^1(\lambda_{10})}$$

$$d'_p = \frac{2(2p+1)}{p(p+1)} k_1 w_1 \omega_2(p, 1) \frac{P_1^1(\lambda_{10})}{Q_1^1(\lambda_{10})} \frac{P_p^1(\lambda_{20})}{Q_p^1(\lambda_{20})}$$

$$(p=2, 3, \ldots \text{ad. inf.}) \qquad \ldots \quad (22)$$

The theory of solutions of the equations of this class has been worked out by Hill,[1] Poincare,[2] Vonkocti,[3] Teoplitz,[4] Hilbert,[5] and others. Hence, A's and B's can be completely evaluated and the problem thus becomes determinate.

3. The complete algebraic values of A's and B's thus determined are not much suitable for numerical calculations. But the constants may be calculated to any degree of approximation as follows:

$$\text{From (7), } \omega_1(m,n)=(-)^n \frac{2^n \lfloor n}{\lfloor 2n+1} \rho_1^{n+1}\Big[D_{t_1}^{\,n} + \frac{\rho_1^2}{2(2n+3)} D_{t_1}^{\,n+2} + \text{etc.}\Big] Q_m(t_1)$$

$$\text{where } \rho_1=\frac{k_2}{k_1},\ t_1=\frac{s}{k_1} \quad = \frac{2^{m+n}\lfloor m \lfloor n \lfloor m+n}{\lfloor 2m+1 \lfloor 2n+1} \, \frac{k_1^m k_2^{n+1}}{s^{m+n+1}}$$

$$\times\Big[1+\frac{\lfloor m+n+2}{2\lfloor m+n}\frac{1}{s^2}\left(\frac{k_1^2}{2m+3}+\frac{k_2^2}{2n+3}\right)+\frac{\lfloor m+n+4}{2\cdot 4\lfloor m+n}\frac{1}{s^4}$$

$$\left\{\frac{k_1^4}{(2m+3)(2m+5)}+\frac{2k_1 k_2^2}{(2m+3)(2n+3)}+\frac{k_2^4}{(2n+3)(2n+5)}\right\}$$

$$+\frac{\lfloor m+n+6}{2\cdot 4\cdot 6\lfloor m+n}\frac{1}{s^6}\left\{\frac{k_1^6}{(2m+3)(2m+5)(2m+7)}\right.$$

$$+\frac{3k_1^4 k_2^2}{(2m+3)(2m+5)(2n+3)}+\frac{3k_1^2 k_2^4}{(2m+3)(2n+3)(2n+5)}$$

$$\left.+\frac{k_2^6}{(2n+3)(2n+5)(2n+7)}\right\}+\text{etc.}\Big] \qquad \ldots \quad (23)$$

(substituting the value of $Q_m(t_1)$ and simplifying).

[1] Hill—" Acta Math. 8, 1-36, 1886.

[2] Poincare—" Bul. Soc. Math. France " 14, 77-90, 1886.

[3] Vonkocti—" Rend. Circ. di Palermo " 28, 255-266, 1909.

[4] Teoplitz— Do. 28, 88-96, 1909.

[5] Hilbert—" Gött. Nachr." p. 157-227, 1906.

Thus the lowest order of ω_1 (m, n) is $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^{m+n+1}$

and therefore the lowest order of $\theta_{p n}$ is $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^{p+n+4}$ putting $m=1$ in (19).

The corresponding expression for ω_2 (m, n) can be readily written down from (23) by interchanging k_1 and k_2.

(A) If the spheroids are so separated that we can neglect the terms of the order $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^{3}$ and higher powers, we have

$$A_1 = k_1\, w_1\, \frac{P_1{}^1(\lambda_{10})}{Q_1{}^1(\lambda_{10})}, \qquad A_p = 0, \ (p=2, 3, 4 \ldots \text{ad. inf.})$$

From symmetry $B_1 = k_2\, w_2\, \dfrac{P_1{}^1(\lambda_{20})}{Q_1{}^1(\lambda_{20})}$, $B_p = 0$ $(p=2, 3, 4 \ldots \text{ad. inf.})$.

$\therefore$ To this order of approximation, we have from (5)

$$\psi = k_1\, w_1\, \frac{P_1{}^1(\lambda_{10})}{Q_1{}^1(\lambda_{10})}\, P_1{}^1(\mu_1)\, Q_1{}^1(\lambda_1)$$

$$+ k_2\, w_2\, \frac{P_1{}^1(\lambda_{20})}{Q_1{}^1(\lambda_{20})}\, P_1{}^1(\mu_2)\, Q_1{}^1(\lambda_2) \qquad \ldots \quad (24)$$

(B) If the terms of the order $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^{5}$ and higher powers are neglected, we have from (17), (18) and (19), (23)

$$A_1 = k_1\, w_1\, \frac{P_1{}^1(\lambda_{10})}{Q_1{}^1(\lambda_{10})} + 3k_2\, w_2\, \omega_1(1, 1)\, \frac{P_1{}^1(\lambda_{10})\, P_1{}^1(\lambda_{20})}{Q_1{}^1(\lambda_{10})\, Q_1{}^1(\lambda_{20})}$$

$$= k_1\, w_1\, \frac{P_1{}^1(\lambda_{10})}{Q_1{}^1(\lambda_{10})} + \tfrac{2}{3}\, k_1\, w_2\, \frac{P_1{}^1(\lambda_{10})\, P_1{}^1(\lambda_{20})}{Q_1{}^1(\lambda_{10})\, Q_1{}^1(\lambda_{20})}\, \frac{k_2{}^3}{s^3}$$

$$A_2 = \tfrac{2}{5}\, k_2\, w_2\, \frac{P_1{}^1(\lambda_{20})\, P_2{}^1(\lambda_{10})}{Q_1{}^1(\lambda_{20})\, Q_2{}^1(\lambda_{10})}\, \frac{k_1{}^2\, k_2{}^2}{s^4}$$

$$A_p = 0, \qquad (p=3, 4, \ldots \text{ad. inf.}).$$

The corresponding expressions for B_1, B_2 can be written down from symmetry

$$B_1 = k_2 w_2 \frac{P_1^1(\lambda_{20})}{Q_1^1(\lambda_{20})} + \frac{3}{5} k_2 w_1 \frac{P_1^1(\lambda_{10})\, P_1^1(\lambda_{20})}{Q_1^1(\lambda_{10})\, Q_1^1(\lambda_{20})} \frac{k_1^3}{s^3}$$

$$B_2 = \frac{3}{5} k_1 w_1 \frac{P_2^1(\lambda_{20})\, P_1^1(\lambda_{10})}{Q_2^1(\lambda_{20})\, Q_1^1(\lambda_{10})} \frac{k_1^2\, k_2^2}{s^4}$$

$$B_p = 0, \qquad (p = 3, 4, \ldots \text{ad. inf.}).$$

Hence, correct to $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^4$, we have

$$v = A_1 P_1^1(\mu_1)\, Q_1^1(\lambda_1) + A_2 P_2^1(\mu_1)\, Q_2^1(\lambda_1)$$
$$+ B_1 P_1^1(\mu_2)\, Q_1^1(\lambda_2) + B_2 P_2^1(\mu_2)\, Q_2^1(\lambda_2) \qquad \ldots \quad (25)$$

where A_1, A_2, B_1, B_2 are given above.

(C) If the terms of the order $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^7$ and higher powers are neglected, we have from (17), (18), (19) and (23)

$$A_1 = k_1 w_1 \frac{P_1^1(\lambda_{10})}{Q_1^1(\lambda_{10})} (1 + \theta_{11})$$
$$+ 3k_2 w_2 \omega_1 (1, 1) \frac{P_1^1(\lambda_{10})\, P_1^1(\lambda_{20})}{Q_1^1(\lambda_{10})\, Q_1^1(\lambda_{20})}$$

$$= k_1 w_1 \frac{P_1^1(\lambda_{10})}{Q_1^1(\lambda_{10})} \left(1 + \frac{4}{9} \frac{P_1^1(\lambda_{10})\, P_1^1(\lambda_{20})}{Q_1^1(\lambda_{10})\, Q_1^1(\lambda_{20})} \frac{k_1^3\, k_2^3}{s^6}\right)$$
$$+ \frac{2}{3} \frac{P_1^1(\lambda_{10})\, P_1^1(\lambda_{20})}{Q_1^1(\lambda_{10})\, Q_1^1(\lambda_{20})} k_1 w_2 \left(1 + \frac{3}{5} \frac{k_1^2 + k_2^2}{s^2}\right) \frac{k_2^3}{s^3}$$

$$A_2 = d_2 = \frac{2}{9} k_2 w_2 \frac{k_1^2\, k_2^2}{s^4} \left\{1 + \frac{10}{s^2}\left(\frac{k_1^2}{7} + \frac{k_2^2}{5}\right)\right\}$$
$$\frac{P_1^1(\lambda_{20})\, P_2^1(\lambda_{10})}{Q_1^1(\lambda_{20})\, Q_2^1(\lambda_{10})}$$

$$A_3 = d_3 = \frac{4}{45} k_2 w_2 \frac{P_3^1(\lambda_{10})\, P_1^1(\lambda_{20})}{Q_3^1(\lambda_{10})\, Q_1^1(\lambda_{20})} \cdot \frac{k_1^3\, k_2^2}{s^5}$$

$$A_4=d_4=\tfrac{4}{105}\,k_2\,w_2\,\frac{P_4^1(\lambda_{10})\,P_1^1(\lambda_{20})}{Q_4^1(\lambda_{10})\,Q_1^1(\lambda_{20})}\cdot\frac{k_1^4\,k_2^2}{s^6}.$$

$$A_p=0\ (p=5,\,6\ldots\text{ad. inf.})$$

The values of B's can be written down from symmetry. Thus

$$B_1=k_2\,w_2\,\frac{P_1^1(\lambda_{20})}{Q_1^1(\lambda_{20})}\left(1+\tfrac{4}{5}\,\frac{P_1^1(\lambda_{10})\,P_1^1(\lambda_{20})}{Q_1^1(\lambda_{10})\,Q_1^1(\lambda_{20})}\,\frac{k_1^3\,k_2^3}{s^6}\right)$$

$$+\frac{2k_1^3\,P_1^1(\lambda_{10})\,P_1^1(\lambda_{20})\,k_2\,w_1}{3\,s^3\,Q_1^1(\lambda_{10})\,Q_1^1(\lambda_{20})}\left(1+\tfrac{6}{5}\,\frac{k_1^2+k_2^2}{s^2}\right).$$

$$B_2=\tfrac{2}{5}\,k_1\,w_1\,\frac{k_1^2\,k_2^2}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_2^2}{7}+\frac{k_1^2}{5}\right)\right\}$$

$$\cdot\,\frac{P_1^1(\lambda_{10})\,P_2^1(\lambda_{20})}{Q_1^1(\lambda_{10})\,Q_2^1(\lambda_{20})}.$$

$$B_3=\tfrac{4}{45}\,k_1\,w_1\,\frac{P_1^1(\lambda_{10})\,P_3^1(\lambda_{20})}{Q_1^1(\lambda_{10})\,Q_3^1(\lambda_{20})}\,\frac{k_1^2\,k_2^3}{s^5}$$

$$B_4=\tfrac{4}{105}\,k_1\,w_1\,\frac{P_1^1(\lambda_{10})\,P_4^1(\lambda_{20})}{Q_1^1(\lambda_{10})\,Q_4^1(\lambda_{20})}\cdot\frac{k_1^2\,k_2^4}{s^6}$$

$$B_5=B_6=\ldots\text{etc.}=0.$$

Hence, correct to $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^6$,

$$v=\sum_{n=1}^{4}A_n\,P_n^1(\mu_1)\,Q_n^1(\lambda_1)+\sum_{n=1}^{4}B_n\,P_n^1(\mu_2)\,Q_n^1(\lambda_2)\quad,\ldots\quad(26)$$

where A's and B's are given above.

In a similar way v can be obtained correct to any order of approximation.

4. *Expressions for v in spherical harmonics.* The solutions (24), (25), (26) etc. can be expressed in spherical harmonics by the help of the following theorems.

From (8), we have, with Hobson's definitions of associated Legendre Functions

$$P_n^{\sigma}(\mu_2)\, Q_n^{\sigma}(\lambda_2) \cos \sigma w$$

$$= \frac{1}{2\pi} \cdot \frac{\lfloor n+\sigma}{\lfloor n-\sigma} \int_{-\pi}^{\pi} Q_n\left(\frac{z_2 + i x_2 \cos u + i y_2 \sin u}{k_2}\right) \cos \sigma u \, du$$

Again[1] $\displaystyle\int_{-\pi}^{\pi} \frac{\cos \sigma u \, d}{(z_2 + i x_2 \cos u + i y_2 \sin u)^{n+1}}$

$$= 2\pi (-1)^{\sigma} \frac{\lfloor n-\sigma}{\lfloor n} \frac{P_n^{\sigma}(\cos \theta_2)}{r_2^{n+1}} \cos \sigma w.$$

with Hobson's definition of P_n^{σ}

Hence, $Q_n^{\sigma}(\lambda_2)\, P_n^{\sigma}(\mu_2) \cos \sigma w$

$$= \frac{\lfloor n+\sigma}{2\pi \lfloor n-\sigma} \cdot \frac{2^n \lfloor n \lfloor n}{\lfloor 2n+1} k_2^{n+1}$$

$$\int_{-\pi}^{\pi} \left\{ \frac{1}{\zeta^{n+1}} + \frac{(n+1)(n+2) k_2^2}{2 \cdot (2n+3)} \frac{1}{\zeta^{n+3}} + \ldots \right\} \cos \sigma u \, du$$

writing ζ_2 for $(z_2 + i x_2 \cos u + i y_2 \sin u)$

$$= (-)^{\sigma} \frac{2^n \lfloor n \lfloor n+\sigma}{\lfloor 2n+1} k_2^{n+1} \left\{ \frac{P_n^{\sigma}(\cos \theta_2)}{r_2^{n+1}} \right.$$

$$\left. + \frac{\lfloor n+2-\sigma}{\lfloor n-\sigma} \cdot \frac{k_2^2}{2 \cdot (2n+3)} \frac{P_{n+2}^{\sigma}(\cos \theta_2)}{r_2^{n+3}} + \text{etc.} \right\} \cos \sigma w.$$

i.e. $Q_n^{\sigma}(\lambda_2)\, P_n^{\sigma}(\mu_2)$

$$= (-)^{\sigma} \frac{2^n \lfloor n \lfloor n+\sigma}{\lfloor 2n+1} k_2^{n+1} \left\{ \frac{P_n^{\sigma}(\cos \theta_2)}{r_2^{n+1}} \right.$$

$$\left. + \frac{\lfloor n+2-\sigma}{\lfloor n-\sigma} \cdot \frac{k_2^2}{2 \cdot (2n+3)} \frac{P_{n+2}^{\sigma}(\cos \theta_2)}{r_2^{n+3}} + \text{etc.} \right\} \qquad \ldots \quad (27)$$

[1] Whittaker-mod. analysis, 15·61. Example, p. 320.

with Hobson's definition of $Q_n^{\sigma}(\lambda_2)$ and Ferrer's definition of $P_n^{\sigma}(\mu)$ and $P_n^{\sigma}(\cos\theta)$ (extracting and cancelling the factor i^{σ} from both the sides).

$$\text{Also}\ \frac{P_p^1(\lambda)}{Q_p^1(\lambda)}=\frac{\dfrac{d}{d\lambda}P_p(\lambda)}{\dfrac{d}{d\lambda}Q_p(\lambda)}=\frac{(2pc_p)^2\,(2p+1)^p}{2^{2p}\,(p+1)\,e^{2p+1}}$$

$$\times\frac{\left\{1-\dfrac{(p-1)(p-2)}{2(2p-1)}e^2+\dfrac{(p-1)(p-2)(p-3)(p-4)}{2\cdot4\cdot(2p-1)(2p-3)}e^4-\text{etc.}\right\}}{\left\{1+\dfrac{(p+2)(p+3)}{2(2p+3)}e^2+\dfrac{(p+2)(p+3)(p+4)(p+5)}{2\cdot4\cdot(2p+3)(2p+5)}e^4+\text{etc.}\right\}}$$

$$\left[\because\lambda=\frac{1}{e}\quad e\ \text{being eccentricity}\right]\qquad\ldots\quad(28)$$

5. From (27) and (28), $k_1\,w_1\,\dfrac{P_1^1(\lambda_{10})}{Q_1^1(\lambda_{10})}\,P_1^1(\mu_1)\,Q_1^1(\lambda_1)$

$$=a_1\,w_1\left[\left(1-\tfrac{3}{5}e_1^2\right)\frac{P_1^1(\cos\theta_1)}{r_1^2}+\frac{a_1^2e_1^2}{5}\,\frac{P_3^1(\cos\theta_1)}{r_1^4}\right]$$

neglecting e^3 and higher powers always.

Similarly, $k_2\,w_2\,\dfrac{P_1^1(\lambda_2)}{Q_1^1(\lambda_{20})}\,P_1^1(\mu_2)\,Q_1^1(\lambda_2)$

$$=a_2^3\,w_2\left[\left(1-\tfrac{3}{5}e_2^2\right)\frac{P_1^1(\cos\theta_2)}{r_2^2}+\frac{a_2^2e_2^2}{5}\,\frac{P_3^1(\cos\theta_2)}{r_2^4}\right]$$

Hence (24) gives

$$v=a_1^3\,w_1\left[\left(1-\tfrac{3}{5}e_1^2\right)\frac{P_1^1(\cos\theta_1)}{r_1^2}+\frac{a_1^2e_1^2}{5}\,\frac{P_3^1(\cos\theta_1)}{r_1^4}\right]$$

$$+a_2^3\,w_2\left[\left(1-\tfrac{3}{5}e_2^2\right)\frac{P_1^1(\cos\theta_2)}{r_2^2}+\frac{a_2^2e_2^2}{5}\,\frac{P_3^1(\cos\theta_1)}{r_2^4}\right]$$

If the equations of the spheroids be written in the form

$$r=a\,[1+\epsilon_1\,P_2\,(\cos\theta_1)]$$

$$r=a'\,[1+\epsilon_2\,P_2\,(\cos\theta_2)]$$

evidently $a=a_1\left(1-\frac{e_1^{2}}{3}\right)$; $a'=a_2\left(1-\frac{e_2^{2}}{3}\right)$,

$$\epsilon_1=\frac{e_1^{2}}{3}\,; \qquad \epsilon_2=\frac{e_2^{2}}{3}\,.$$

Also, $a_1^{3}\,(1-\frac{2}{5}\,e_1^{2})=\dfrac{5-\epsilon_1}{5+2\epsilon_1}\,a^{3}$ correct to ϵ_1.

$a_2^{3}\,(1-\frac{2}{5}\,e_2^{2})=\dfrac{5-\epsilon_2}{5+2\epsilon_2}\,a'^{3}$ correct to ϵ_2.

$$\therefore v=a^{3}\,w_1\,\frac{5-\epsilon_1}{5+2\epsilon_1}\,\frac{P_1{}^{1}\,(\cos\theta_1)}{r_1^{2}}+\frac{3}{5}\,a^{5}\,\epsilon_1\,w_1\,\frac{P_3{}^{1}\,(\cos\theta_1)}{r_1^{4}}$$

$$+a'^{3}\,w_2\,\frac{5-\epsilon_2}{5+2\epsilon_2}\,\frac{P_1{}^{1}\,(\cos\theta_2)}{r_2^{2}}+\frac{3}{5}\,a'^{5}\,\epsilon_2\,w_2\,\frac{P_3{}^{1}\,(\cos\theta_2)}{r_2^{4}} \qquad \ldots \quad (29)$$

Under the above conditions *viz.* neglecting ϵ_1^{2} and higher powers and $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^{3}$ and higher powers, this problem has been attempted in a previous issue of this Bulletin.[1] The solution for v obtained there does not contain the 2nd and 4th terms of the right hand side of (29). This is due to the omission, through mistake, of the co-efficients of $\dfrac{P_3{}^{1}\,(\cos\theta_1)}{r_1^{4}}$ and $\dfrac{P_3{}^{1}\,(\cos\theta_2)}{r_2^{4}}$ correct to the above orders of approximation,—which is also obvious from the author's own own calculations of those co-efficients correct to higher approximation given in his results (43) and (44) from which the co-efficients of $\dfrac{P_3{}^{1}\,(\cos\theta_1)}{r_1^{4}}$ and $\dfrac{P_3{}^{1}\,(\cos\theta_2)}{r_2^{4}}$ are respectively

$$\frac{3w_1\,a^{5}\,\epsilon_1}{5-8\epsilon_1}-\frac{w_2\,a^{3}\,a'^{3}\,(5-\epsilon_2)}{s^{3}\,(5+2\epsilon_2)}\cdot\frac{\epsilon_1}{5-8\epsilon_1}$$

and

$$\frac{3\,w_2\,a'^{5}\,\epsilon_2}{5-8\epsilon_2}-\frac{w_1\,a'^{3}\,a^{3}}{s^{3}}\cdot\frac{5-\epsilon_1}{5+2\epsilon_1}\cdot\frac{\epsilon_2}{5-8\epsilon_2}$$

[1] Bul. Cal. Math. Soc. Vol. X, No. 1 ibid.

From these two the co-efficients correct to above orders of approximation are respectively $\frac{3\,w_1\,a^5\,\epsilon_1}{5}$ and $\frac{3\,w_2\,a'^5\,\epsilon_2}{5}$ which are identical with those obtained by me in (29). These cannot evidently be neglected in the expression for v correct to orders of approximation already referred to.

6. Again from (25), $A_1\,P_1^{\,1}(\mu_1)\,Q_1^{\,1}(\lambda_1)$

$$=\left\{k_1\,w_1\,\frac{P_1^{\,1}(\lambda_{10})}{Q_1^{\,1}(\lambda_{10})}+\tfrac{2}{3}\,k_1\,w_2\,\frac{k_2^{\,3}\,P_1^{\,1}(\lambda_{10})\,P_1^{\,1}(\lambda_{20})}{s^3\,Q_1^{\,1}(\lambda_{10})\,Q_1^{\,1}(\lambda_{20})}\right\}$$

$$P_1^{\,1}(\mu_1)\times Q_1^{\,1}(\lambda_1)$$

$$=a_1^{\,3}\left\{(1-\tfrac{4}{5}\,e_1^{\,2})\,w_1-\frac{a_2^{\,3}}{s^3}\,w_2\,(1-\tfrac{4}{5}\,e_1^{\,2})\,(1-\tfrac{4}{5}\,e_2^{\,2})\right\}\frac{P_1^{\,1}(\cos\theta_1)}{r_1^{\,2}}$$

$$+\frac{a_1^{\,5}\,e_1^{\,2}}{5}\,\left(w_1-\frac{a_2^{\,3}}{s^3}\,w_2\right)\frac{P_3^{\,1}(\cos\theta_1)}{r_1^{\,4}}$$

neglecting $e_1^{\,3}$ and higher powers.

Also, proceeding similarly $A_2\,P_2^{\,1}(\mu_1)\,Q_2^{\,1}(\lambda_1)$

$$=-\frac{a_1^{\,5}\,a_2^{\,3}\,w_2}{s^4}\,(1-\tfrac{10}{7}\,e_1^{\,2})\,(1-\tfrac{4}{5}\,e_2^{\,2})\,\frac{P_2^{\,1}(\cos\theta_1)}{r_1^{\,3}}$$

$$-\tfrac{8}{7}\,\frac{a_1^{\,7}\,a_2^{\,3}}{s^4}\,w_2\,\frac{P_4^{\,1}(\cos\theta_1)}{r_1^{\,5}}\,e_1^{\,2}$$

Similarly writing down the expressions for $B_1\;P_1^{\,1}\;(\mu_2)\;Q_1^{\,1}\;(\lambda_2)$ and $B_2\;P_2^{\,1}\;(\mu_2)\;Q_2^{\,1}\;(\lambda_2)$ from symmetry, we have from (25),

$$v=a_1^{\,3}\left\{(1-\tfrac{4}{5}\,e_1^{\,2})\,w_1-\frac{a_2^{\,3}}{s^3}\,w_2\,(1-\tfrac{4}{5}\,e_1^{\,2})\,(1-\tfrac{4}{5}\,e_2^{\,2})\right\}\frac{P_1^{\,1}(\cos\theta_1)}{r_1^{\,2}}$$

$$-\frac{a_1^{\,5}\,a_2^{\,3}}{s^4}\,w_2\,(1-\tfrac{10}{7}\,e_1^{\,2})\,(1-\tfrac{4}{5}\,e_2^{\,2})\,\frac{P_2^{\,1}(\cos\theta_1)}{r_1^{\,3}}$$

$$+\frac{a_1^{\,5}\,e_1^{\,2}}{5}\,\left(w_1-\frac{a_2^{\,3}}{s^3}\,w_2\right)\frac{P_3^{\,1}(\cos\theta_1)}{r_1^{\,4}}-\tfrac{8}{7}\,\frac{a_1^{\,7}\,a_2^{\,3}}{s^4}\,w_2$$

$$\frac{P_4^{\,1}(\cos\theta_1)}{r_1^{\,5}}\,e_1^{\,2}$$

$$+a_2^3\left\{\left(1-\tfrac{6}{5}e_2^2\right)w_2-\frac{a_1^3}{s^2}w_1\left(1-\tfrac{6}{5}e_1^2\right)\left(1-\tfrac{6}{5}e_2^2\right)\right\}\frac{P_1^1(\cos\theta_2)}{r_2^2}$$

$$-\frac{a_1^3a_2^5}{s^4}w_1\left(1-\tfrac{10}{7}e_2^2\right)\left(1-\tfrac{6}{5}e_1^2\right)\frac{P_2^1(\cos\theta_2)}{r_2^3}$$

$$+\frac{a_2^5e_2^2}{5}\left(w_2-\frac{a_1^3}{s^3}w_1\right)\frac{P_3^1(\cos\theta_2)}{r_2^4}-\tfrac{3}{7}\frac{a_1^3a_2^7}{s^4}w_1$$

$$\frac{P_4^1(\cos\theta_2)}{r_2^5}e_2^2 \qquad \ldots \quad (30)$$

Then remembering that ϵ_1^2, $\epsilon_1\epsilon_2$, ϵ_2^2 and higher powers are always to be neglected, the co-efficient of $\dfrac{P_1^1(\cos\theta_1)}{r_1^2}=w_1a^3\dfrac{5-\epsilon_1}{5+2\epsilon_1}$

$$-w_2\frac{a^3a'^3}{s^3}\;\frac{5-\epsilon_1}{5+2\epsilon_1}\cdot\frac{5-\epsilon_2}{5+2\epsilon_2}.$$

This differs from the corresponding expression of Mr. Dutt which does not contain the last term, the omission of which is also due to a mistake in approximation on his part. This omission has, as I shall presently shew, affected the correctness of Mr. Dutt's expression for co-efficient of $\dfrac{P_3^1(\cos\theta_1)}{r_1^4}$ as well.

The co-efficient of $\dfrac{P_2^1(\cos\theta_1)}{r_1^3}$

$$=-\frac{a_1^5a_2^3}{s^4}w_2\left(1-\tfrac{10}{7}e_1^2\right)\left(1-\tfrac{6}{5}e_2^2\right).$$

$$=-\frac{a^5a'^3w_2}{s^4}\;\frac{5-\epsilon_2}{5+2\epsilon_2}\left(1+\frac{5\epsilon_1}{7-3\epsilon_1}\right)$$

The co-efficient of $\dfrac{P_4^1(\cos\theta_1)}{r_1^5}=-\tfrac{3}{7}\;\dfrac{a_1^7a_2^3w_2e_1^2}{s^4}$

$$=-\tfrac{6}{7}\epsilon_1\frac{a^7a'^3}{s^4}w_2$$

The last two results are identical with those obtained by Mr. Dutt.

But the co-efficient of $\dfrac{P_3{}^1(\cos\theta_1)}{r_1{}^4} = \dfrac{a_1{}^5 e_1{}^2}{5}\left(w_1 - \dfrac{a_2{}^3}{s^3} w_2\right)$

$$= \frac{3a^5 w_1 \epsilon_1}{5-8\epsilon_1} - \frac{3a^5 a'^3}{s^3} w_2 \frac{\epsilon_1}{5-8\epsilon_1} \cdot \frac{5-\epsilon_2}{5+2\epsilon_2}.$$

($\because \epsilon_1{}^2, \epsilon_1 \epsilon_2$ etc. and higher powers are to be neglected).

The corresponding expression obtained by Mr. Dutt is

$$\frac{3w_2 a^5 e_1}{5-8\epsilon_1} - \frac{w_2 a^5 a'^3}{s^3} \cdot \frac{5-\epsilon_2}{5+2\epsilon_2} \cdot \frac{\epsilon_1}{5-8\epsilon}.$$

The reason for this difference is to his wrongly omitting the term of the order $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^3$ in the co-efficient of $\dfrac{P_1{}^1(\cos\theta_1)}{r_1{}^2}$. For, according to his notation, the equation (42) giving the co-efficient A_3 is

$$w_1 a^2 \epsilon_1 = \frac{A_3}{a^3}(5-8\epsilon_1) - \frac{2A_1}{a}\epsilon_1 + \epsilon_1 \frac{a^2}{s^2} \cdot R_2$$

If A_1 be wrongly taken equal to $w_1 a^3 \dfrac{5-\epsilon_1}{5+2\epsilon_1}$, $R_2 = 2\dfrac{w_2 a'^3}{5} \cdot \dfrac{5-\epsilon_2}{5+2\epsilon_2}$, A_3 is obtained in the form obtained by Mr. Dutt. But if on the other hand we take the value of the co-efficient correct to $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^3$ which cannot of course be neglected correct to the order of approximation already referred to in the Case (B) *i.e.* if in his equation (42),

$$A_1 = w_1 a^3 \frac{5-\epsilon_1}{5+2\epsilon_1} - w_2 \frac{a^3 a'^3}{s^3} \frac{5-\epsilon_1}{5+2\epsilon_1} \cdot \frac{5-\epsilon_2}{5+2\epsilon_2}, \text{ we obtain}$$

$$A_3 = \frac{3a^5 w\epsilon_1}{5-8\epsilon_1} - \frac{3 w_2 a^5 a'^3}{s^3} \cdot \frac{\epsilon_1}{5-8\epsilon_1} \cdot \frac{5-\epsilon_2}{5+2\epsilon_2}, \text{ which is the value}$$

obtained by me.

Thus from his own equation, we get the correct value of the co-efficient of $\frac{P_3{}^1(\cos\theta_1)}{r_1{}^4}$ by using the correct expression for the co-efficient of $\frac{P_1{}^1(\cos\theta_1)}{r_1{}^2}$.

In a similar way other expressions for v may be expressed in spherical harmonics by the help of (27) and (28).

TWO OBLATE SPHEROIDS ROTATING ABOUT THEIR COMMON AXIS OF REVOLUTION.

7. Let $x_1 = k'_1\,(1-\mu_1{}^2)^{\frac{1}{2}}\,(\lambda_1{}^2+1)^{\frac{1}{2}}\cos w$.

$$y_1 = k'_1\,(1-\mu_1{}^2)^{\frac{1}{2}}\,(\lambda_1{}^2+1)^{\frac{1}{2}}\sin w;$$

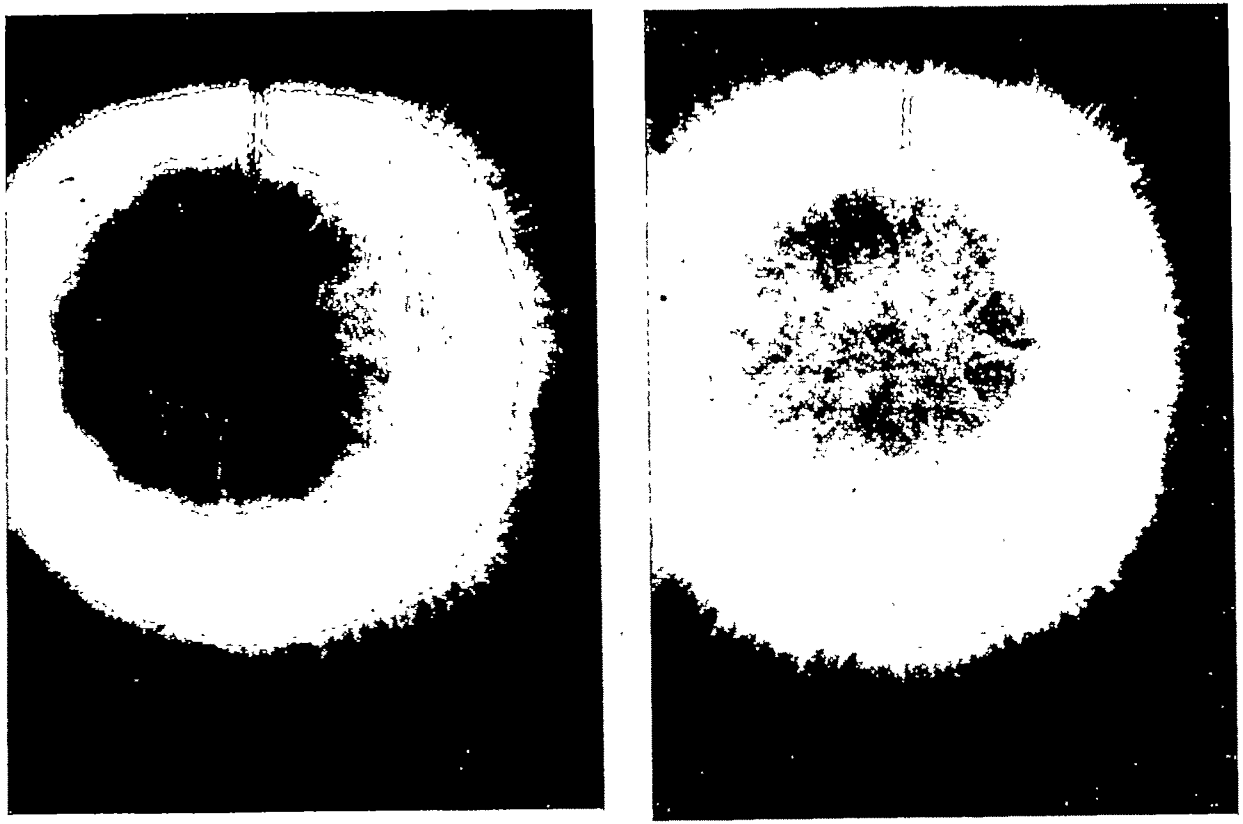
$$z_1 = k'_1\,\mu_1\,\lambda_1;$$

$$x_2 = k'_2\,(1-\mu_2{}^2)^{\frac{1}{2}}\,(\lambda_2{}^2+1)^{\frac{1}{2}}\cos w$$

$$y_2 = k'_2\,(1-\mu_2{}^2)^{\frac{1}{2}}\,(\lambda_2{}^2+1)^{\frac{1}{2}}\sin w$$

$$z_2 = k'_2\,\mu_2\,\lambda_2$$

be the two systems of co-ordinates referred to the centres of the two spheroids as origin and (λ_1, μ_1, w), (λ_2, μ_2, w) be the two systems of planetary spheroidal co-ordinates. It is easy to see that if $\frac{k'_1}{i}$ be written for k_1 and $i\lambda$ for λ in the prolate spheroidal co-ordinates, the corresponding expressions for oblate spheroidal co-ordinates are readily deducible. Hence, angular velocities etc. remaining same, the expressions for v in the present case are at once obtained by writing $\frac{k'}{i}$ and $i\lambda$ for k and λ respectively in the corresponding expressions for v in prolate spheroidal case discussed before.



On Caustics formed by Diffraction.

[To face page 35.]

4

On Caustics formed by Diffraction

BY

Panchanan Das, M.Sc.

From the examination of various diffraction plates of the Fresnel class, Prof. C. V. Raman[1] and Dr. S. K. Mitra[2] came to the conclusion that these patterns exhibit a marked concentration of luminosity along curves agreeing generally in position and form with the evolute of the shadow, that should be formed according to geometrical optics, of the diffracting boundary. They also observed a series of fringes running parallel to the curves of maximum luminosity and found in fact that these curves presented a marked similarity with the caustics formed by reflection and refraction. The new class of caustics here arising may be referred to as deffraction caustics. The object of the present paper is to discuss the theory of the formation of these caustics and to place the same on an exact mathematical basis. In order to illustrate the subject, a photograph[3] of the diffraction-pattern of an one-anna coin, which has an undulating edge, was also taken, which showed these caustics beautifully. By assuming the equation of the boundary of the coin aforesaid, to be of the form

$$r=a\ (1+\epsilon \cos n\theta),$$

a calculation of the distance between consecutive fringes was carried out and it was found to agree fairly with that actually measured from the plates.

The analysis is based on a paper by Rubinowicz,[4] the substance of which may, for convenience, be reproduced here.

All diffraction problems lead to an equation of the form .—

$$\nabla^2\ u+k^2\ u=0.$$

[1] Phys Rev. 13. 1919

[2] Phil Mag. July, 1919.

[3] This was first done by Dr S. K. Mitra, loc cit

[4] Ann d Physik Bd 53. pp 257-1917

Let $u\ (x, y, z)$ be a finite, differentiable solution of the above equation. Then the surface-integral,

$$\frac{1}{4\pi}\int_{(G)}\int\left\{\bar{u}\,\frac{\partial}{\partial n}\left(\frac{e^{\iota\kappa r}}{r}\right)-\frac{e^{\iota\kappa r}}{r}\cdot\frac{\partial \bar{u}}{\partial n}\right\}df \qquad \ldots \quad (1)$$

over any region vanishes or equals u, according as the pole of r is without or within the region G, where $\bar{u}$ and $\frac{\partial \bar{u}}{\partial n}$ are values on the surface of G.

Kirch-hoff assumes that on the shaded side of the screen S, the light-disturbance is

$$u=\frac{1}{4\pi}\int_{(F)}\int\left\{\bar{u}\,\frac{\partial}{\partial n}\left(\frac{e^{\iota\kappa r}}{r}\right)-\frac{e^{\iota\kappa r}}{r}\cdot\frac{\partial \bar{u}}{\partial n}\right\}df \qquad \ldots \quad (2)$$

where F is the diffracting aperture.

If the source of light L, (fig. 1) be a point-source then $u=\frac{e^{\iota\kappa\rho}}{\rho}$ where ρ is the distance of a point on F from L. Let K be the surface of the shadow-cone due to F bounded by the line B.

$L\,ds = \rho_s$

$L\,d\sigma = \rho$

$P\,ds = r_s$

$P\,d\sigma = r.$

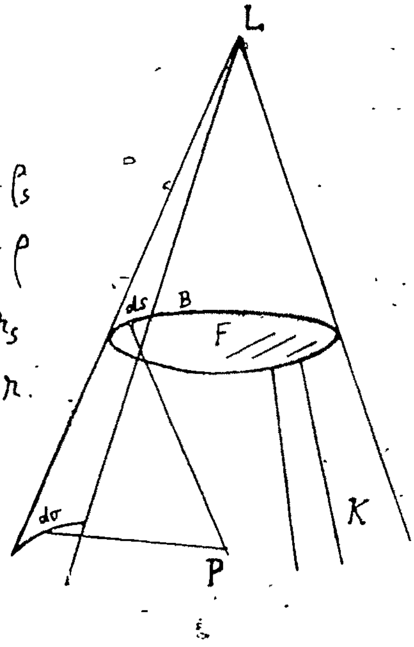


Fig 1

Then the integral (1) extending over F and K, becomes the discontinuous function, u_E given by

$$u_E=\frac{1}{4\pi}\int_{(F+K)}\int\left\{\frac{e^{\iota\kappa\rho}}{\rho}\cdot\frac{\partial}{\partial n}\left(\frac{e^{\iota\kappa r}}{r}\right)-\frac{\partial}{\partial n}\left(\frac{e^{\iota\kappa\rho}}{\rho}\right)\cdot\frac{e^{\iota\kappa r}}{r}\right\}df \qquad \ldots \quad (3)$$

which equals $\frac{e^{\iota\kappa\rho}}{\rho}$ in the region (directly illuminated) bounded by F and K, and vanishes outside.

So, u_E ("incident" u) is the disturbance of geometrical optics, which neglects diffraction.

Thus, on the surface K, $\frac{\partial}{\partial n}\left(\frac{e^{\iota\kappa\rho}}{\rho}\right)=0$, and

$$\frac{\partial}{\partial n}\left(\frac{e^{\iota\kappa r}}{r}\right)=\frac{\partial}{\partial r}\left(\frac{e^{\iota\kappa r}}{r}\right)\cos(n, r)$$

$$=\left(\frac{\iota\kappa}{r}-\frac{1}{r^2}\right)e^{\iota\kappa r}\cos(n, r).$$

So from (2) and (3) we get for Kirch-hoffs' diffraction integral, the expression

$$u=u_E-\frac{1}{4\pi}\int_K\int\frac{e^{\iota\kappa(r+\rho)}}{\rho}\cdot\left(\frac{\iota\kappa}{r}-\frac{1}{r^2}\right)\cos(n, r)\,df \quad \dots \quad (4)$$

This is transformed into a line integral round the edge E. Call it u_B="diffracted" u. For orthogonal surface coordinates, let us take ρ, and the section line σ of K with spheres ρ=const. If a linear element ds of B is distant ρ_s from L, then (fig. 1),

$$d\sigma=\frac{\rho}{\rho_s}\sin(\rho_s, ds)\,ds.$$

$$\therefore\quad df=d\rho d\sigma=\frac{\rho d\rho}{\rho_s}\sin(\rho_s, ds)\,ds.$$

If r_s be the distance of any point P from ds, we have

$$r^2=r_s^2+(\rho-\rho_s)^2+2r_s(\rho-\rho_s)\cos(r_s, \rho_s).$$

also, $\cos(n, r) = \frac{r_s}{r} \cos(u, n_s)$, Thus,

$$u_B = -\frac{1}{4\pi} \int_B ds. \sin(\rho_s, ds) \cos(n, r_s).$$

$$\frac{r_s}{\rho_s} \int_{\rho_s}^{\infty} d\rho\, e^{\iota\kappa(\rho+r)} \left(\frac{\iota\kappa}{r^2} - \frac{1}{r^3} \right).$$

$$\text{Now} \int_{\rho_s}^{\infty} e^{\iota\kappa(\rho+r)} \frac{\iota\kappa}{r^2} d\rho = \int_{\rho_s}^{\infty} \frac{d}{d\rho} \cdot \left\{ e^{\iota\kappa(\rho+r)} \right\} \cdot \frac{d\rho}{\left(1+\frac{dr}{d\rho}\right) r^2}$$

$$= \int_{\rho_s}^{\infty} \frac{d}{d\rho} \cdot \left\{ e^{\iota\kappa(\rho+r)} \right\} \cdot \frac{d\rho}{\{r+\rho-\rho_s+r_s \cos(r_s, \rho_s)\} \cdot r},$$

$$\text{and } \frac{d}{d\rho} \cdot \frac{1}{\{r+\rho-\rho_s+r_s \cos(r_s, \rho_s)\} r} = -\frac{1}{r^3}.$$

$$\text{Thus,} \int_{\rho_s}^{\infty} e^{\iota\kappa(\rho+r)} \left(\frac{\iota\kappa}{r^2} - \frac{1}{r^3} \right) d\rho$$

$$= \int_{\rho_s}^{\infty} \frac{d}{d\rho} \left\{ e^{\iota\kappa(\rho+r)} \frac{1}{[r+\rho-\rho_s+r_s \cos(r_s, \rho_s)] r} \right\} d\rho.$$

$$= -\frac{e^{\iota\kappa(\rho_s+r_s)}}{r_s^2 [1+\cos(r_s, \rho_s)]}.$$

Thus, $u = u_E + u_B$

$$= u_E + \frac{1}{4\pi} \int_B \frac{e^{\iota\kappa\rho_s}}{\rho_s} \cdot \frac{e^{\iota\kappa r_s}}{r_s} \cdot \frac{\cos(n, r_s) \sin(\rho_s, ds)}{\{1+\cos(r_s, \rho_s)\}} \quad \ldots \quad (5)$$

The diffraction wave due to the element ds is then

$$du_B = \frac{1}{4\pi} \cdot \frac{e^{\iota\kappa\rho_s}}{\rho_s} \cdot \frac{e^{\iota\kappa r_s}}{r_s} \cdot \frac{\cos(n, r_s)}{1+\cos(r_s, \rho_s)} \sin(\rho_s, ds)\, ds \quad \ldots \quad (6)$$

We shall suppose that both the source and the screen are at large distances compared with the maximum diameter of the diffracting aperture or obstacle. In order to evaluate the integral of (6) for the most general boundary we shall have to effect some reductions and simplifications.

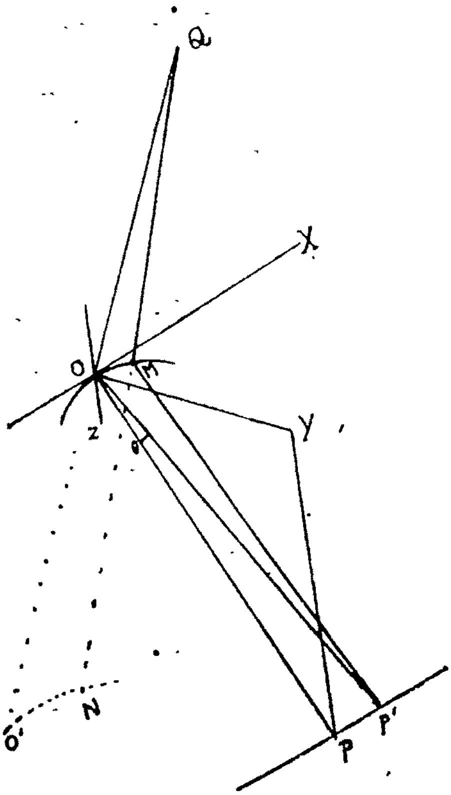


Fig 2.

Let the equation of the boundary referred to the tangent and normal as axes, at any point O, be

$$y=bx^2+cx^3$$

If the screen be held parallel to the aperture at a distance h, the coordinates of any point P on the line of intersection of the screen and normal plane to the boundary curve through O, are (O, R, h) say.

Through P draw PP′ in the plane of the screen parallel to the tangent OX. Let the angle POP′$=\theta$, where θ is small. Then we have,

$$OP^2=R^2+h^2=l^2 \text{ and } OP=l, \text{ say.}$$

Then $OP'=l\sec\theta$, and $PP'=l\tan\theta=l\theta$, approximately.

The coordinates of M are $(x, y, 0)$; those of P are $(l\theta, R, h)$. We also have

$$MP'^2=(x-l\theta)^2+(R-y)^2+h^2$$

$$=(x-l\theta)^2+(R-bx^2-cx^3\ldots)^2+h^2$$

$$=l^2+(x-l\theta)^2-2R\ (bx^2+cx^3\ldots)$$

$$\therefore MP'=l\left\{1+\frac{(x-l\theta)^2-2R\ (bx^2+cx^3)}{2l^2}\right\} \text{ approximately.}$$

Let the source of light be Q, (x_0, y_0, z_0).

Then $OQ^2=(x_0{}^2+y_0{}^2+z_0{}^2)=L^2$, say.

And $MQ^2=(x_0-x)^2+(y_0-y)^2+z_0{}^2$

$=L^2-2xx_0-2y_0(bx^2+cx^3)+x^2$. neglecting higher powers of x than the cube.

Hence $MQ=L\{1+\dfrac{x^2-2y_0\ (bx^2+cx^3)-2xx_0}{2L^2}\}$.

Therefore, $\qquad MQ+MP'$

$$=l+L+\frac{1}{2l}\ \{x^2-2l\theta x+l^2\theta^2-2R\ (bx^2+cx^3)\}$$

$$+\frac{1}{2L}\ \{x^2-2xx_0-2y_0\ (bx^2+cx^3)\}.$$

With an object in view which will be seen later on, we equate the coefficients of x and x^2 to zero; thus:—

$$\theta=-\frac{x_0}{L} \qquad \ldots \quad (7)$$

and

$$\frac{1-2\,Rb}{l}+\frac{1-2y_0b}{L}=0 \qquad \ldots \quad (8)$$

Thus the point P′ $(l\theta, R, h)$ is determined uniquely with reference to the origin 0 by the equations (7) and (8). If we now regard 0 as a variable point on the boundary the corresponding point P′ describes a curve locus. We shall first show that this curve locus is approximately the evolute of the geometrical shadow of the boundary.

Let the point N (x', y', z') be the point on the geometrical shadow of the boundary corresponding to the point M $(x, y, 0)$ and let $0'$ correspond to the origin 0, on the boundary; then we have

$$\frac{x-x'}{x-x_0}=\frac{y-y'}{y-y_0}=\frac{z'}{z_0}=-\frac{h}{z_0}.$$

Eliminating x, y with the help of the equation

$$y=bx^2+cx^3+\ldots, \text{ we get}$$

$$y'\ z_0+hy_0=b\ \frac{(x'\ z_0+hx_0)^2}{z_0+h}+c\ \frac{(x'\ z_0+hx_0)^3}{(z_0+h)^2}+\ldots$$

Substituting $x'=\zeta-\frac{hx_0}{z_0}$, and $y'=\eta-\frac{hy_0}{z_0}$, we get

$$\eta=\frac{bz_0}{z_0+h}\;\xi^2+\frac{cz_0^{\,2}}{(z_0+h)^2}\,\zeta^3+\qquad\ldots\quad(9)$$

This is the equation of the geometrical shadow of the boundary referred to parallel axes on the screen, of which the origin is $(-\frac{hx_0}{z_0},-\frac{hy_0}{z_0},h)$ with reference to original axes. The form of (9) suggests at once, that the ξ—and η—axes are the tangent and normal to the geometrical shadow of the boundary at O′.

Let the coordinates of the centre of curvature at O′ referred to ζ—and η—axes be $(0, R_1)$, where R_1=radius of curvature at O′. Then from (9), we easily see that

$$R_1=\frac{1}{2b}\left(1+\frac{h}{z_0}\right).$$

Hence referred to the original axes, the coordinates of this centre of curvature are

$$\left(-\frac{hx_0}{z_0},\ \frac{1}{2b}+\frac{h}{2bz_0}-\frac{hy_0}{z_0},\ h\right).$$

Now, if we regard x_0, y_0 as small compared with z_0, so that $L=z_0$, and $l=h$, approximately these coordinates might be written :—

$$\left(-\frac{lx_0}{L},\ \frac{1}{2b}+\frac{l}{L}\cdot\frac{L}{2b}-\frac{l}{L}\cdot y_0,\ h\right).$$

Calculating the values of θ and R from (7) and (8) respectively, we see that the point P′ given by $(l\theta, R, h)$ and the centre of curvature at O′ are identical. Hence, the locus described by the point P′ as the origin is shifted, is the evolute of the geometrical shadow of the boundary.

We now proceed to show that the locus of P′ traces also the general outline of the region of maximum intensity in the diffraction pattern.

The light-disturbance du from an element ds of the boundary is given by (6). Let us discuss in detail and calculate the value of du for the elements of the boundary near O at the point P′.

If we expand du in terms of the element ds or rather x, and retain only a few terms of the expansion, we shall find that the change in the periodic factor $e^{\iota\kappa(\rho_s+r_s)}$ will be extremely rapid, while the

changes in the other factors are negligible, as x varies slowly. Thus we shall look upon $e^{\iota\kappa(\rho_s+r_s)}$ as the only variable factor and the rest as constant.

Now let $\rho_s+r_s=a_0+a_1x+a_2x^2+a_3x^3+\ldots$

Then since $k=\frac{2\pi}{\lambda}$, where λ is the wave-length we have,

$$k(\rho_s+r_s)=2\pi\left(\frac{a_0}{\lambda}+a_1\cdot\frac{x}{\lambda}+a_2\cdot\frac{x^2}{\lambda}+a_3\cdot\frac{x^3}{\lambda}+\ldots\right).$$

If x or x^2 be of the same order of magnitude as λ, then the above expression changes extremely rapidly with x, and thus $e^{\iota\kappa(\rho_s+r_s)}$, being an oscillating function, will, on being integrated with reference to x, have in general an inappreciable value. But if $a_1=a_2=0$, then $\frac{x^2}{\lambda}$ and the higher powers become negligible, so that $e^{\iota\kappa(\rho_s+r_s)}$ will have a constant value correct to the order of $\frac{x^3}{\lambda}$, and the integral will have an appreciable value. This fact has an important bearing on the formation of diffraction patterns.

If P′, instead of being uniquely determined with reference to the origin by (7) and (8), were any point chosen at random near P′ as determined by (7) and (8) then the constants a_1, a_2 would not in general be zero, and hence du would not have an appreciable value at P′. But if P′ is determined by (7) and (8), the value of du is appreciable there and is in fact a maximum. Hence the locus traced by P′ as the origin is shifted will be one of maximum illumination in the diffraction pattern. But we have seen above that the locus of P′ is the evolute of the geometrical shadow of the boundary approximately. Hence the diffraction pattern exhibits a strong illumination along the evolute of the shadow of the boundary of the diffracting aperture or obstacle.

We next give a theory of the diffraction caustics that are formed close to the prominent lines of the pattern.

Instead of determining the point P′ uniquely with reference to the origin, let us now abolish the condition (7) but retain (8), so that the position of the point P′ is a function of the small angle θ occurring in fig. 1. Then we can write du thus:—

$$du=\mathrm{A}\,e^{\iota\kappa(\rho_s+r_s)}\,ds$$

$$=A\,e^{\iota\kappa\,(MQ+MP')}\,ds$$

$$=A\,e^{\iota\kappa\left\{l+L+\frac{l\theta^2}{2}-x\left(\theta+\frac{x_0}{L}\right)-cx^3\left(\frac{R}{l}+\frac{y_0}{L}\right)\right\}}\,dx$$

$$=A'\,e^{\alpha x+\beta x^3}\,dx \qquad \ldots \quad (10)$$

In order to integrate this we assume, as is generally done, that the influences of the elements of the boundary at an appreciable distance from 0 annul each other, and integrate the expression between $\pm\infty$. Thus,

$$u=A\int_{-\infty}^{\infty} e^{\alpha x+\beta x^3}\,dx \qquad \ldots \quad (11)$$

This is the well known Airy integral, and its properties have been discussed by Mascart.[1] He shows that as a is varied a series of brilliant fringes appear, separated by dark intervals. Obviously a is a linear function of θ and varying θ means studying the light-disturbance in the neighbourhood of the main diffraction pattern.

An examination of the boundary of the shadow of the coin shows that the part of the waving contour between two successive points of inflexion may ba taken as a small arc of an ellipse. The part of the diffraction pattern corresponding to this element is easily seen to have a cusp just like the evolute of an ellipse.

In order to have a rough estimate of the spacing of the caustics we may assume that the equation of the boundary of the coin is of the form :—

$$r=a\,(1+\epsilon\cos n\theta)$$

where ϵ is small and $n=12$.

To determine ϵ, the maximum and minimum diameters of the coin were measured by means of callipers and the values of a and ϵ were calculated from these values. Thus,

$$2a\,(1+\epsilon)=20{\cdot}6\ mm.$$

$$2a\,(1-\epsilon)=19{\cdot}5\ mm.$$

whence $\qquad a=10{\cdot}02\ mm$, and $\epsilon={\cdot}027$ approximately.

[1] Traite d'Optique. Tome I, pp. 393-4.

Now in order to evaluate the integral (11) for the above boundary we have first to find the equation of the curve with reference to the tangent and normal at any point on the boundary as axes.

Now we may regard the curve as a superposition of the displacements y_1 and y_2, where y_1 as a function of x represents the equation of a small arc of a circle with reference to axes parallel to the tangent and normal at any point and y_2 as a function of x represents the usual harmonic curve with proper dimensions Thus let $y_1+f=\frac{1}{2a}(x+g)^2$, and $y_2=-a\epsilon\cos n\left(\frac{x}{a}+\theta_0\right)$.

$$\text{Then } y=y_1+y_2=-f+\frac{1}{2a}(x^2+2gx+g^2)$$

$$-a\epsilon\left\{\cos n\theta_0\left(1-\frac{n^2x^2}{2a^2}\right)-\sin n\theta_0\left(\frac{nx}{a}-\frac{n^3x^3}{6a^3}\right)\right\}.$$

$$=-f+g^2-a\epsilon\cos n\theta_0+\left(\frac{g}{a}+n\epsilon\sin n\theta_0\right)x+\frac{1}{2a}(1+n^2\epsilon\cos n\theta_0)x^2$$

$$-\frac{n^3\epsilon\sin n\theta_0}{6a^2}x^3$$

Now let the values of f and g be so adjusted that the absolute term and the term in x vanish.

Thus $y=bx^2+cx^3$,

where $$b=\frac{1}{2a}(1+\epsilon n^2\cdot\cos n\theta_0)$$

and $$c=-\frac{\epsilon n^3\sin n\theta_0}{6a^2}.$$

The integral, we have to study, is

$$\int e^{-\iota\kappa\left\{\left(\theta+\frac{x_0}{L}\right)x+c\left(\frac{R}{l}+\frac{y_0}{L}\right)x^3\right\}}dx,$$

or taking the real part only, the expression under the integral sign, is

$$\cos\left\{k\left(\theta+\frac{x_0}{L}\right)x+kc\left(\frac{R}{l}+\frac{y_0}{L}\right)x^3\right\}.$$

Put $$\frac{\pi}{2}u^3=kc\left(\frac{R}{l}+\frac{y_0}{L}\right)x^3$$

and $$\frac{\pi}{2}zu=k\left(\theta+\frac{x_0}{L}\right)x.$$

Then the above expression becomes $\cos\frac{\pi}{2}(u^3+zu)$, which form has been exhaustively studied in Mascart's book. It follows that

$$z=\frac{\theta+\frac{x_0}{L}}{\sqrt[3]{\frac{\pi^2}{4k^2}\left(\frac{R}{l}+\frac{y_0}{L}\right)}}$$

We remember that $k=\frac{2\pi}{\lambda}$ and it follows from (8) that

$$\frac{R}{l}+\frac{y_0}{L}=\frac{1}{2b}\left(\frac{1}{l}+\frac{1}{L}\right).$$

Thus

$$z=\frac{\theta+\frac{x_0}{L}}{\sqrt[3]{\frac{c\lambda^2}{32b}\left(\frac{1}{l}+\frac{1}{L}\right)}}$$

If the first two maxima correspond to the values θ_1, θ_2 and z_1, z_2 respectively, we have

$$z_1-z_2=\frac{\theta_1-\theta_2}{\sqrt[3]{\frac{c\lambda^2}{32b}\left(\frac{1}{l}+\frac{1}{L}\right)}}$$

If d is the linear distance between these two maxima, then $\theta_1-\theta_2=\frac{d}{l}$, approximately.

The values of z_1 and z_2 (Mascart) corresponding to these values are 5·14 and 3·47 respectively. From an examination of the pattern it

was estimated that the point on the boundary for which the value d was actually measured corresponded to the value $n\theta_0=45°$ approximately. If we now measure everything in millimetres, then

$$\lambda \text{ for yellow light} = \cdot 00058 mm.$$

$$l=1330,\ L=2050,\ b=\cdot 3,\ c=\cdot 36.$$

Substitution of these in the formula readily gives

$$d=\cdot 56\ mm. \text{ approx.}$$

An actual measurement of d was carried out by means of a travelling microscope and the mean value was about ·6*mm*., thus giving a fairly good agreement between calculated and observed values

My best thanks are due to Professor C. V. Raman for his continual help and active interest in this paper.

5

The Generalized Angle Concept

BY

Dr. Philip Franklin.

In a paper entitled "On the Angle-Concept in n-dimensional Geometry"[1] S. Ganguly suggested the problem of studying the inclinations of two k-spaces in an n-space $(k<n)$, and gave a complete discussion for the case $k=2$. In this paper we shall extend his results to the general case, and incidentally develop some re'ations concerning the the volumes in the k-spaces.

We shall obtain our results by the use of tensors, and shall accordingly recall the necessary definitions and elementary properties. In a Euclidean n-space, the vectors drawn from a fixed point form a set linearly dependent on n independent ones. We start with one such set $(e_1 \ldots e_n)$ and call them the unit vectors, although they need not be of the same length or at right angles to one another when interpreted through a Cartesian system in the ordinay way. A second set of unit vectors $(\bar{e}_1 \ldots \bar{e}_n)$ will be related to the first set by equations of the form :

$$(1) \qquad \bar{\ } = \sum_k A^k_i e_k \text{ or } A^k_i e_k$$

as we shall in future omit the summation sign, it being understood for all indices which appear twice. To obtain the new components of a vector x^i we express it as a linear combination of the e's :

$$(2) \qquad x^i e_i$$

and apply (1). This gives :

$$(3) \qquad x^i = A^i_k \bar{x}^k \text{ or } \bar{x}^k = \breve{A}^k_i x^i$$

where the matrix of the $\breve{A}$'s is the inverse of that of the A's. The transformation (3) is said to be *contragredient* with that of (1), and the

[1] Bulletin of the Calcutta Mathematical Society, Vol. IX, 1918, p. 11.

x^i's are accordingly called *contravariant* variables. The e_i's, or any variables y_i transformed by the equations :

$$(4) \qquad y_i = \breve{A}_i^k \bar{y}_k \text{ or } \bar{y}_i = A_i^k y_k$$

transformed *cogrediently* with (1) are called *covariant* variables.

If a set of functions p_{kr}^{ij}, constants in any one coordinate system, and defined for all coordinate systems is given, which are such that the expression

$$(5) \qquad p_{kr}^{ij} x_i y_j z^k u^r$$

is the same in all coordinate systems, where x and y are any covariant variables, and z and u are any contravariant variables, the set of p's are the *components* of a *tensor*, *contravariant* in i and j (in general in the superscripts) and *covariant* in k and r (in general in the subscripts). (5) refers to a tensor of the fourth order; the generalization to n-indices is obvious. Since (5) is an invariant, if it has a geometrical significance in one system, in terms of that given to the variables (*e.g.*, we may regard the contravariant ones as components of vectors, the covariant ones will be interpreted presently) it will maintain its significance in all systems.

From our definition of contravariance, it follows that

$$(6) \qquad x^i y_i$$

is an invariant, and hence that the (contravariant) components of a vector are the components of a tensor of the first order. As a tensor of the second order, consider the length of a vector. As we are dealing with oblique coordinates in Euclidean space, its square will be a quadratic form : (which may be taken as a symmetric form) :

$$(7) \qquad g_{ij} x^i x^j.$$

This will give rise to a bilinear form :

$$(8) \qquad g_{ij} x^i y^j$$

the "scalar product" of the vectors x and y. Thus the g_{ij} are the components of a tensor, whose significance in terms of the two vectors used is the product of their lengths by the cosine of the angle between them. If we form

$$(9) \qquad x_i = g_{ij} x^j \text{ and } \bar{x}_i = \bar{g}_{ij} \bar{x}^j$$

in view of (8) we see that the x_i are covariant variables, the covariant components of the vector x^i : by solving (9), we obtain :

$$(10)\quad x^i = g^{ij} x_j \;(\|g^{ij}\| \text{ the inverse of } \|g_{ij}\|)$$

enabling us to calculate contravariant components from covariant ones.

There are several methods of forming new tensors from given tensors. Thus the sum of two tensors of the same order with corresponding indices is a tensor. ($e.g.$, $p^{ij}+q^{ij}=t^{ij}$). Likewise the product of any two tensors is a tensor ($e.g.$, $p^{ij}q_r=t_r^{ij}$). These facts are easily proved by noting that the expression for t to be shown an invariant is the sum or product of the invariants for p and q. Further the results obtained from a given tensor by equating a covariant and a contravariant index and summing for this index (contraction) are the components of a tensor, ($e.g.$, $p_r^{ijr}=t^{ij}$). For $t^{ij}x_i y_j = p_r^{ijr}x_i y_j$ is invariant since both $p_r^{ijr} x_i y_j z_r u^r$ and $z_r u^r$ are invariant. We shall use these methods to build up, from g_{ij} and vectors an invariant which bears the same relation to k-space that the scalar product does to the vector, and shall obtain our results from this invariant.

Let a k-space be determined by the k independent vectors $a^i, b^i \ldots q^i$. The product of these will be a tensor, and by permuting the indices and adding or subtracting the results, we find that :

$$(11)\quad K^{ij\ldots s} = \begin{vmatrix} a^i & a^j & \cdots & a^s \\ \vdots & \vdots & & \vdots \\ q^i & q^j & \cdots & q^s \end{vmatrix}$$

is a tensor. Since

$$(12)\quad K^{ij\ldots s} x_i y_j \ldots u_s = K^{ij\ldots s} g_{im} g_{jn} \ldots g_{st} x^m y^n \ldots u^t$$

is invariant, we may obtain a new tensor :

$$(13)\quad K_{mn\ldots t} = K^{ij\ldots s} g_{im} g_{jn} \ldots g_{st}$$

and since from (11) $K^{ij\ldots s}$ is zero if two of the indices are equal, and changes sign when two are interchanged ($i.e.$, is skew-symmetric), we may write in place of (13) :

$$(14)\quad k!\,K_{mn\ldots t} = \begin{vmatrix} a^i & a^j & \cdots & a^s \\ \vdots & \vdots & & \vdots \\ q^i & q^j & & q^s \end{vmatrix} \begin{vmatrix} g_{im} & g_{jm} & \cdots & g_{sm} \\ \vdots & \vdots & & \vdots \\ g_{it} & g_{jt} & \cdots & g_{st} \end{vmatrix}$$

By multiplying the tensors given in (11) and (14), and contracting with respect to all the indices, we obtain the invariant :

$$(15)\quad k!V^2_{a-q}=\begin{vmatrix} a^m & a^n & \dots & a^t \\ \vdots & \vdots & & \vdots \\ q^m & q^n & \dots & q^t \end{vmatrix} \begin{vmatrix} a^i & a^j & \dots & a^s \\ \vdots & \vdots & & \vdots \\ q^i & q^j & \dots & q^s \end{vmatrix} \begin{vmatrix} g_{im} & g_{jm} & \dots & g_{sm} \\ \vdots & \vdots & & \vdots \\ g_{it} & g_{jt} & \dots & g_{st} \end{vmatrix}$$

V_{a-q} *is the volume of the k-dimensional parallelopiped formed from the* vectors $a^i \dots q^i$. For if we use a Cartesian coordinate system (for which $g_{ij}=\delta_{ij}=1$ or 0 according as $i=$ or $\neq j$) whose first k axes are in the k-space determined by $a^i \dots q^i$, (15) reduces to the single term :

$$(16)\qquad k!\begin{vmatrix} a^1 & a^2 & \dots & a^k \\ \vdots & \vdots & & \vdots \\ q^1 & q^2 & \dots & q^k \end{vmatrix}^2 \begin{vmatrix} 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{vmatrix}$$

after the terms resulting from the different permutations are combined, and the first determinant is the expression for the volume of the k-parallelopiped in Cartesian coordinates, $k+1$ of whose vertices are the origin, and the extremities of the vectors $a^i \dots q^i$ drawn from the origin.[1]

For two k-spaces, one given by $a^i, b^i \dots q^i$ and the other by $A^i, B^i \dots Q^i$ we may form the expression (11) for the first, and (14) for the second, and by multiplication and contraction form the expression

$$(17)\quad k!S_{a,A}=\begin{vmatrix} a^m & a^n & \dots & a^t \\ \vdots & \vdots & & \vdots \\ q^m & q^n & \dots & q^t \end{vmatrix} \begin{vmatrix} A^i & A^j & \dots & A^s \\ \vdots & \vdots & & \vdots \\ Q^i & Q^j & \dots & Q^s \end{vmatrix} \begin{vmatrix} g_{im} & g_{jm} & \dots & g_{sm} \\ \vdots & \vdots & & \vdots \\ g_{it} & g_{jt} & \dots & g_{st} \end{vmatrix}$$

analagous to (15), which is the generalization of the scalar product. If we write

$$(18)\qquad \cos\Omega=\frac{S_{a,A}}{V_{a-q}V_{A-Q}}$$

$\cos\Omega$ is independent of the particular set of vectors used to fix the k-spaces, only depending on these k-spaces. For, replacing one of the $a^i \dots q^i$ by a linear combination of them will merely multiply the determinants involving these components by a factor, which will appear once in both V_{a-q} and $S_{a,A}$ and hence cancel out. But by such

[1] For a simple proof of this formula for n-space see the Mathematical Gazette, Vol. X, 1921, p. 324.

replacements we can go from any set of vectors $a^i \ldots q^i$ to any other set determining the same k-space. Ω is the "angle of projectivity" between the k-spaces.

If we use the Cartesian coordinates previously introduced, which makes the first one of the coordinate k-spaces, the numerator of (18) will reduce to a single term, containing V_{a-q} as a factor, and on cancelling this out, we shall have :

$$(19)\qquad \cos\Omega = \frac{\begin{vmatrix} A^1 & A^2 & \ldots & A^k \\ \vdots & \vdots & & \vdots \\ Q^1 & Q^2 & \ldots & Q^k \end{vmatrix}}{V_{A-Q}}$$

which shows that $\cos\Omega$ is the ratio of the volume in the $a^i \ldots q^i$ $(1 \ldots k)$ space formed by projecting the volume V_{A-Q}, to this last volume; where by projecting the volume we mean taking the components of the vectors used to form V_{A-Q} in the $a^i \ldots q^i$ space for the projected volume. Since we are using Cartesian coordinates, the denominator is ;

$$(20)\qquad V_{A-Q} = \sqrt{\Sigma \begin{vmatrix} A^i & A^j & \ldots & A^s \\ \vdots & \vdots & & \vdots \\ Q^i & Q^j & \ldots & Q^s \end{vmatrix}^2}$$

(Each combination counts only once in the summation.)

as follows from (15) since $g_{ij} = \delta_{ij}$. This shows that $\cos\Omega < 1$ also that if we formed expressions analogous to (19) for all the coordinate k-spaces ("direction cosines" of the k-space A—Q) the sum of their squares would be unity.

Finally, if we take two k-spaces given by sets of vectors of unit length $(m^1 \ldots m^k)$ and $(n^1 \ldots n^k)$, and determine the maximum or minimum values of the angle θ between two lines, one in each space; using a method entirely analogous to that given by Ganguly,[1] we obtain the equation (in terms summed for i, $0 < i \leq n$):

$$(21)\qquad \begin{vmatrix} \cos\theta & m^1_i m^2_i \cos\theta & \ldots & m^1_i m^k_i \cos\theta & m^1_i n^1_i & \ldots & m^1_i n^k_i \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ m^k_i m^1_i \cos\theta & m^k_i m^2_i \cos\theta & \ldots & m^k_i m^k_i \cos\theta & m^k_i n^1_i & \ldots & m^k_i n^k_i \\ n^1_i m^1_i & n^1_i m^2_i & \ldots & n^1_i m^k_i & \cos\theta & \ldots & n^1 m^k \cos\theta \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ n^k_i m^1_i & n^k_i m^2_i & \ldots & n^k_i m^k_i & n^k_i n^1_i \cos\theta & \ldots & \cos\theta \end{vmatrix} = 0$$

[1] l. c., p. 16.

which shows that the product of the k values of $\cos^2\theta$ is:

$$(22)\qquad \frac{|\, m^s_i \; n^t_i \,|^2}{|\, m^s_i \; m^t_i \,| \; |\, n^s_i \; n^t_i \,|}$$

in which we have merely written the term in the sth row and tth column in each determinant. But in Cartesian coordinates, (17) gives: on replacing $a'\ldots q'$ by $m^1_i \ldots m^k_i$ and $A' \ldots Q'$ by $n^1_i \ldots n^k_i$:

$$(23)\qquad S_{n-k} = \sum |\, m^s_{j_t} \,| \; |\, n^s_{j_t} \,| = |\, m^s_i \; n^t_i \,|$$

where the summation extends over all combinations of k integers out of n $(j_1 \ldots j_k)$ each combination counted once, and the second equality follows from a well-known matrix identity.[1] This shows that (22) is equivalent to $S^2_{n-k}/V^2_k V'^2_k = \cos^2\Omega$, and hence that the angle of projectivity of two k-spaces is the product of the k extremal values of $\cos\theta$, θ being the angle between a pair of lines one in each space.

We have thus shown that the invariant S, which is a function of two sets of k-vectors expressed by (17) in oblique coordinates, and by (23) in Cartesian coordinates is the generalization of the scalar product, being equal to the product of the volumes of the two k-parallelopipeds constructed on the two sets of vectors, by the cosine of the angle of projectivity of the two k-spaces determined by them. The cosine of the angle of projectivity of two k-spaces is equal to the ratio of the volume of a k-parallelopiped in one of the spaces to that of its projection in the other space: it is also equal to the product of k extremal values of the cosine of the angle between a pair of lines, one from each of the k-spaces.

[1] Scott and Mathews, Determinants, pp. 50-51.

6

On the motion of two Spheroids in an infinite liquid

BY

Nripendranath Sen, M.Sc.

The first writer to attempt the problem of the motion of two spheroids or ellipsoids in an infinite liquid is Prof. Karl Pearson[1] whose method does not, however, admit of further development and does not, therefore, lead to the complete solution of the problem. In a previous issue of this Bulletin,[2] Dr Bibhutibhusan Datta attempted the problem of motion of two spheriods of small eccentricities in an infinite liquid along their common axis of revolution. In a recent issue of the American Journal of Mathematics,[3] Dr. Datta has solved the more general case of the same problem *viz.*, the motion of two spheroids of any eccentricities in an infinite liquid along their common axis of revolution.

The object of the present paper is to present the solution of a much more difficult problem *viz.*, the problem of the motion of an infinite liquid due to arbitrary movement of two speriods, both prolate and oblate, having a common axis of revolution. The problem has been completely solved first for two prolate spheriods, having any velocity of translation together with any velocity of rotation with no limitation regarding their ellipticities and central distance, and the solution for the case of two oblate speroids has been deduced therefrom by suitable substitutions.

I have shown that Dr. Datta's results of the problem referred to above may be deduced as a particular case of the general problem discussed in the present paper.

[1] Karl Pearson—"On the motion of spherical and ellipsoidal bodies in fluid media Part II. Quart. Journ. Math. Vol. 20.

[2] Bibhutibhushan Datta, D.Sc.—"On the motion of two spheroids in an infinite liquid along the common axis of revolution" Bul. Cal Math. Soc., Vol. 7, pp. 49-60.

[3] Bibhutibhushan Datta, D.Sc.—"On the motion of two spheroids in an infinite liquid along their common axis of revolution," Americ. Journ. Math., Vol. 43, pp. 31-42, 1921.

MOTION OF TWO PROLATE SPHEROIDS HAVING A COMMON AXIS OF REVOLUTION.

2. Let 0_1, 0_2 be centres of the two spheroids, $0_1\,0_2$ the common axis of revolution taken as z—axis and let the two systems of co-ordinates referred to parallel axes at 0_1 and 0_2 be

$$x_1=\kappa_1(1-\mu_1^2)^{\frac{1}{2}}\;(\lambda_1^2-1)^{\frac{1}{2}}\cos\omega\;;\qquad x_2=\kappa_2(1-\mu_2^2)^{\frac{1}{2}}\;(\lambda_2^2-1)^{\frac{1}{2}}\cos\omega$$

$$y_1=\kappa_1(1-\mu_1^2)^{\frac{1}{2}}\;(\lambda_1^2-1)^{\frac{1}{2}}\sin\omega\;;\qquad y_2=\kappa_2(1-\mu_2^2)^{\frac{1}{2}}\;(\lambda_2^2-1)^{\frac{1}{2}}\sin\omega$$

$$z_1=\kappa_1\mu_1\lambda_1\;;\qquad z_2=\kappa_2\mu_2\lambda_2$$

where (λ_1,μ_1,ω), (λ_2,μ_2,ω) are the two systems of plolate spheroidal coordinates so that $\lambda_1=\lambda_{10}, \lambda_2=\lambda_{20}$ on the surfaces of the given spheroids at 0_1 and 0_2 whose semi-axes are $a_1,c_1(a_1>c_1)$ and $a_2,c_2(a_2>c_2)$ respectively, whose eccentricities are e_1,e_2 respectively and central distance $0_1 0_2=s$. Let (u_1,v_1,w_1,p_1,q_1) and (u_2,v_2,w_2,p_2,q_2) be the components of motion of the two spheroids, whose rotations about z—axis have not been taken into account here owing to the fact that there will be no motion of the liquid due to such rotations. To find the liquid motion due to such motions of the spheroids.

The problem before us is, therefore, to find a velocity potential ϕ satisfying the following conditions *viz.*,

$$\nabla^2\phi=0 \qquad \dots \quad (1)$$

$$\phi=0 \text{ at infinity } i.e. \text{ when } \lambda=\infty \qquad \dots \quad (2)$$

$$\frac{\partial\phi}{\partial\lambda_1}=-\left\{\left(u_1\frac{\partial x_1}{\partial\lambda_1}+v_1\frac{\partial y_1}{\partial\lambda_1}+w_1\frac{\partial z_1}{\partial\lambda_1}\right)+p_1\left(y_1\frac{\partial z_1}{\partial\lambda_1}-z_1\frac{\partial y_1}{\partial\lambda_1}\right)\right.$$

$$\left.+q_1\left(z_1\frac{\partial x_1}{\partial\lambda_1}-x_1\frac{\partial z_1}{\partial\lambda_1}\right)\right\}=-\kappa_1 w_1 P_1(\mu_1)-\left\{\kappa_1 u_1\frac{\lambda_1 P^1{}_1(\mu_1)}{(\lambda_1^2-1)^{\frac{1}{2}}}\right.$$

$$\left.+\frac{\kappa_1^2 q_1 P_2^1(\mu_1)}{3(\lambda_1^2-1)^{\frac{1}{2}}}\right\}\cos\omega-\left\{\kappa_1 v_1\frac{\lambda_1}{(\lambda_1^2-1)^{\frac{1}{2}}}P^1{}_1(\mu_1)\right.$$

$$\left.-\frac{\kappa_1^2 p_1}{3(\lambda_1^2-1)^{\frac{1}{2}}}P_2^1(\mu_1)\right\}\sin\omega \text{ when } \lambda_1=\lambda_{10} \qquad \dots \quad (3)$$

and $$\frac{\partial\phi}{\partial\lambda_2}=-\kappa_2\omega_2 P_1(\mu_2)-\left\{\kappa_2 u_2\frac{\lambda_2 P^1{}_1(\mu_2)}{(\lambda_2^2-1)^{\frac{1}{2}}}\right.$$

$$\left.+\frac{\kappa_2^2 q_2 P_2^1(\mu_2)}{3(\lambda_2^2-1)^{\frac{1}{2}}}\right\}\cos\omega-\left\{\kappa_2 v_2\frac{\lambda_2 P_1^1(\mu_2)}{(\lambda_2^2-1)^{\frac{1}{2}}}-\frac{\kappa_2^2 p_2 P_2^1(\mu_2)}{3(\lambda_2^2-1)^{\frac{1}{2}}}\right\}\sin\omega$$

when $$\lambda_2=\lambda_{20} \qquad \dots \quad (4)$$

3. Assume

$$\phi=\sum_{n=1}^{\infty}\{A_n P_n(\mu_1)Q_n(\lambda_1)+a_n P_n(\mu_2)Q_n(\lambda_2)$$

$$+P_n^1(\mu_1)Q_n^1(\lambda_1)(B_n\cos\omega+C_n\sin\omega)$$

$$+P_n^1(\mu_2)Q_n^1(\lambda_2)(b_n\cos\omega+c_n\sin\omega)\} \quad \ldots \quad (5)$$

Evidently this value of ϕ satisfies (1) and (2). Now, to determine A's, B's, C's etc., so as to satisfy the boundary conditions (3) and (4).

It has been proved in a previous paper[1] of the author that

$$P_n^{\sigma}(\mu_2)Q_n^{\sigma}(\lambda_2)=(-)^{\sigma}\frac{\lfloor n+\sigma}{\lfloor n-\sigma}\sum_{m=\sigma}^{\infty}(2m+1)\frac{\lfloor m-\sigma}{\lfloor m+\sigma}\omega_1(m,n)$$

$$\times P_m^{\sigma}(\mu_1)P_m^{\sigma}(\lambda_1) \quad \ldots \quad (6)$$

for all $+ve$ integral values of σ including zero,

where $\omega_1(m,n)=\dfrac{(-)^n 2^n \lfloor n}{\lfloor 2n+1}\rho_1^{n+1}\left[D_{t_1}^n+\dfrac{\rho_1^2}{2\cdot(2n+3)}D_{t_1}^{n+2}+\ldots\right]Q_m(t_1)$

where $\quad \rho_1^2=\dfrac{k_2^2}{k_1^2}=\dfrac{a_2^2e_2^2}{a_1^2e_1^2}$, $t_1=\dfrac{s}{k_1}$, $D_{t_1}^n=\dfrac{d^n}{dt_1^n}$ $\quad \ldots \quad (7)$

Substituting the value of $P_n^{\sigma}(\mu_2)Q_n^{\sigma}(\lambda_2)$ when $\sigma=0$, in (5) from (6) we have

$$\phi=\sum_{n=1}^{\infty}\left\{A_n P_n(\mu_1)Q_n(\lambda_1)+a_n\sum_{m=1}^{\infty}(2m+1)\omega_1(m,n)P_m(\mu_1)P_m(\lambda_1)\right.$$

$$+P_n^1(\mu_1)Q_n^1(\lambda_1)(B_n\cos\omega+C_n\sin\omega)$$

$$\left.-(b_n\cos\omega+c_n\sin\omega)\sum_{m=1}^{\infty}\frac{n(n+1)(2m+1)}{m(m+1)}\omega_1(m,n)P_m^1(\mu_1)P_m^1(\lambda_1)\right\}$$

[1] Nripendranath Sen—"On the steady motion of a viscous fluid due to the rotation of two spheroids." "Bul. Cal. Math. Soc." *Present issue.* Results (6), (8), (9), (10) and (11).

Hence from the surface condition (3), we have,

$$-k_1 w_1 P_1(\mu_1) - \left\{ k_1 u_1 \frac{\lambda_{10} P_1^1(\mu_1)}{(\lambda_{10}^2-1)^{\frac{1}{2}}} + \frac{k_1^2 q_1 P_2^1(\mu_1)}{3(\lambda_{10}^2-1)^{\frac{1}{2}}} \right\} \cos\omega$$

$$- \left\{ k_1 v_1 \frac{\lambda_{10} P_1^1(\mu_1)}{(\lambda_{10}^2-1)^{\frac{1}{2}}} - \frac{k_1^2 p_1 P_2^1(\mu_1)}{3(\lambda_{10}^2-1)^{\frac{1}{2}}} \right\} \sin\omega$$

$$= \sum_{n=1}^{\infty} \left\{ A_n P_n(\mu_1) Q'_n(\lambda_{10}) + a_n \sum_{m=1}^{\infty} (2m+1)\omega_1(m,n) P'_m(\lambda_{10}) P_m(\mu_1) \right.$$

$$+ P^1_n(\mu_1) Q'^1_n(\lambda_{10})(B_n \cos\omega + C_n \sin\omega)$$

$$\left. -(b_n \cos\omega + c_n \sin\omega) \sum_{m=1}^{\infty} \frac{n(n+1)(2m+1)}{m(m+1)} \omega_1(m,n) P'^1_m(\lambda_{10}) P^1_m(\mu_1) \right\}$$

at every point of the spheroid $\lambda_1 = \lambda_{10}$ *i.e.* for all values of μ_1 and ω.

$\therefore$ Equating the co-efficients of $\cos\omega$, $\sin\omega$, and co-efficients of P_n's and P_n^1's, we have

$$-k_1 w_1 = A_1 Q'_1(\lambda_{10}) + 3P'_1(\lambda_{10}) \sum_{n=1}^{\infty} a_n \omega_1(1,n) \quad \cdots \quad (8)$$

$$0 = A_p Q'_p(\lambda_{10}) + (2p+1) P'_p(\lambda_{10}) \sum_{n=1}^{\infty} a_n \omega_1(p,n)$$

$$(p=2,3,\cdots ad\ inf.) \quad \cdots \quad (9)$$

$$-k_1 u_1 \frac{\lambda_{10}}{(\lambda_{10}^2-1)^{\frac{1}{2}}} = B_1 Q'^1_1(\lambda_{10}) - \frac{3}{2} P'^1_1(\lambda_{10}) \sum_{n=1}^{\infty} n(n+1)\omega_1(1,n) b_n \quad (10)$$

$$-\frac{k_1^2 q_1}{3(\lambda_{10}^2-1)^{\frac{1}{2}}} = B_2 Q'^1_2(\lambda_{10}) - \frac{5}{6} P'^1_2(\lambda_{10}) \sum_{n=1}^{\infty} n(n+1)\omega_1(2,n) b_n \quad (11)$$

$$0 = B_p Q'^1_p(\lambda_{10}) - \frac{(2p+1) P'^1_p(\lambda_{10})}{p(p+1)} \sum_{n=1}^{\infty} n(n+1)\omega_1(p,n) b_n \quad (12)$$

$$(p=3,4,\ldots ad\ inf.)$$

$$-k_1 v_1 \frac{\lambda_{10}}{(\lambda_{10}^2-1)^{\frac{1}{2}}} = C_1 Q'^1_1(\lambda_{10}) - \tfrac{3}{2} P'^1_1(\lambda_{10}) \sum_{n=1}^{\infty} n(n+1)\omega_1(1,n) c_n \quad (13)$$

$$-\frac{k_1^2 p_1}{3(\lambda_{10}^2-1)^{\frac{1}{2}}}=C_2 Q'^{1}_{2}(\lambda_{10})-\tfrac{5}{6}P'^{1}_{2}(\lambda_{10})\sum_{n=1}^{\infty} n(n+1)\omega_1(2,n)c_n \quad (14)$$

$$0=C_p Q'^{1}_{p}(\lambda_{10})-\frac{(2p+1)}{p(p+1)}P'^{1}_{p}(\lambda_{10})\sum_{n=1}^{\infty} n(n+1)\omega_1(p,n)c_n \ldots \quad (15)$$

$$(p=3,4,\ldots ad\ inf.)$$

The corresponding equations giving a's, b's, c's can be written down from symmetry from the above equations or from the boundary condition (4). Thus,

$$-k_2 w_2=a_1 Q'_1(\lambda_{20})+3\sum_{n=1}^{\infty} A_n\omega_2(1,n) \quad \ldots \quad (16)$$

$$0=a_p Q'_p(\lambda_{20})+(2p+1)P'_p(\lambda_{20})\sum_{n=1}^{\infty} A_n\omega_2(p,n) \quad (17)$$

$$(p=2,3\ldots ad\ inf.)$$

$$-k_2 u_2\frac{\lambda_{20}}{(\lambda_{20}^2-1)^{\frac{1}{2}}}=b_1 Q'^{1}_{1}(\lambda_{20})-\tfrac{3}{2}P'^{1}_{1}(\lambda_{20})\sum_{n=1}^{\infty} n(n+1)\omega_2(1,n)B_n \quad (18)$$

$$-\frac{k_2^2 q_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}}}=b_2 Q'^{1}_{2}(\lambda_{20})-\tfrac{5}{6}P'^{1}_{2}(\lambda_{20})\sum_{n=1}^{\infty} n(n+1)\omega_2(2,n)B_n \ldots \quad (19)$$

$$0=b_p Q'^{1}_{p}(\lambda_{20})-\frac{(2p+1)P'^{1}_{p}(\lambda_{20})}{p(p+1)}\sum_{n=1}^{\infty} n(n+1)\omega_2(p,n)B_n \quad (20)$$

$$(p=3,4\ldots ad\ inf.)$$

$$-k_2 v_2\frac{\lambda_{20}}{(\lambda_{20}^2-1)^{\frac{1}{2}}}=c_1 Q'^{1}_{1}(\lambda_{20})-\tfrac{3}{2}P'^{1}_{1}(\lambda_{20})\sum_{n=1}^{\infty} n(n+1)\omega_2(1,n)C_n \quad (21)$$

$$\frac{k_2^2 p_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}}}=c_2 Q'^{1}_{2}(\lambda_{20})-\tfrac{5}{6}P'^{1}_{2}(\lambda_{20})\sum_{n=1}^{\infty} n(n+1)\omega_2(2,n)C_n \quad (22)$$

$$0=c_p Q'^{1}_{p}(\lambda_{20})-\frac{(2p+1)}{p(p+1)}P'^{1}_{p}(\lambda_{20})\sum_{n=1}^{\infty} n(n+1)\omega_2(p,n)C_n \quad \ldots \quad (23)$$

$$(p=3,4,\ldots ad\ inf.)$$

$$\text{where } \omega_2(m,n) = \frac{(-)^n 2^n \lfloor n}{\lfloor 2n+1} \rho_2^{\,n+1} \Big[D^n_{t_2} + \frac{\rho_2^2}{2\cdot(2n+3)} D^{n+2}_{t_2} + \text{etc.} \Big] P_m(t_2) \quad \dots \quad (24)$$

where $\rho_2 = \dfrac{k_1}{k_2},\ t_2 = \dfrac{s}{k_2},\ D^n_{t_2} = \dfrac{d^n}{dt_2^n}$ etc

The equations (8) to (24) are sufficient to determine sets of unknown constants A's, B's, C's etc., as will be shewn presently.

To determine A's and a's, subtitute the values of a's in (8) and (9) from (16) and (17). We, then, have after a little simplification,

$$A_1 - \sum_{n=1}^{\infty} \theta_{1n} A_n = -\frac{k_1 w_1}{Q'_1(\lambda_{10})} + \frac{3\omega_1(1,1)}{Q'_1(\lambda_{10}) Q'_1(\lambda_{20})} k_2 \omega_2 \quad \dots \quad (25)$$

$$A_p - \sum_{n=1}^{\infty} \theta_{pn} A_n = (2p+1) \frac{P'_p(\lambda_{10}) \omega_1(p,1)}{Q'_p(\lambda_{10}) Q'_1(\lambda_{10})} k_2 \omega_2 \quad \dots \quad (26)$$

$(p=2,3,\dots ad\ inf.)$

$$\text{where } \theta_{pn} = (2p+1) \frac{P'_p(\lambda_{10})}{Q'_p(\lambda_{10})} \sum_{m=1}^{\infty} (2m+1) \frac{P'_m(\lambda_{20})}{Q'_m(\lambda_{20})} \omega_1(p,m) \omega_2(m,n) \quad (27)$$

The equations giving a's can be found out independently or may be written from (25), (26) etc. Thus from symmetry,

$$a_1 - \sum_{n=1}^{\infty} \theta'_{1n} a_n = -\frac{k_2 w_2}{Q'_1(\lambda_{20})} + \frac{3\omega_2(1,1)}{Q'_1(\lambda_{20}) Q'_1(\lambda_{10})} k_1 w_1 \quad \dots \quad (28)$$

$$a_p - \sum_{n=1}^{\infty} \theta'_{pn} a_n = (2p+1) \frac{P'_p(\lambda_{20}) \omega_2(p,1)}{Q'_p(\lambda_{20}) Q'_1(\lambda_{10})} k_1 w_1 \quad \dots \quad (29)$$

$(p=2,3,\dots ad\ inf.)$

$$\text{where } \theta'_{pn} = (2p+1) \frac{P'_p(\lambda_{20})}{Q'_p(\lambda_{20})} \sum_{m=1}^{\infty} (2m+1) \frac{P'_m(\lambda_{10})}{Q'_m(\lambda_{10})} \omega_2(p,m) \omega_1(m,n) \quad (30)$$

To find B's and b's, substitute the values of b's in (10) and (11) from (18), (19) and (20), we have, after a little simplification,

$$B_1-\sum_{n=1}^{\infty}\phi_{1n}B_n=-\frac{\lambda_{10}k_1u_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}$$

$$-\frac{3P'^1_1(\lambda_{10})\lambda_{20}\omega_1(1,1)k_2u_2}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}-\frac{3P'^1_1(\lambda_{10})\omega_1(1,2)k_2^2q_2}{Q'^1_1(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})} \quad (31)$$

$$B_2-\sum_{n=1}^{\infty}\phi_{2n}B_n=-\frac{k_1^2q_1}{3(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{10})}$$

$$-\frac{5}{3}\,\frac{P'^1_2(\lambda_{10})\lambda_{20}\omega_1(2,1)k_2u_2}{Q'^1_2(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}-\frac{5}{3}\,\frac{P'^1_2(\lambda_{10})\omega_1(2,2)k_2^2q_2}{Q'^1_2(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})} \quad (32)$$

$$B_p-\sum_{n=1}^{\infty}\phi_{pn}B_n=-\frac{2(2p+1)P'^1_p(\lambda_{10})}{p(p+1)Q'^1_p(\lambda_{10})}\times\left[\frac{\lambda_{20}\omega_1(p,1)k_2u_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}+\frac{\omega_1(p,2)k_2^2q_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}\right]\ldots \quad (33)$$

$$(p=3,4\cdots ad\ inf.)$$

where $$\phi_{pn}=\frac{(2p+1)P'^1_p(\lambda_{10})}{p(p+1)Q'^1_p(\lambda_{10})}$$

$$\sum_{m=1}^{\infty}n(n+1)(2m+1)\omega_1(p,m)\omega_2(m,n)\frac{P'^1_m(\lambda_{20})}{Q'^1_m(\lambda_{20})} \quad \ldots \quad (34)$$

Also from symmetry,

$$b_1-\sum_{n=1}^{\infty}\phi'_{1n}b_n=-\frac{\lambda_{20}k_2u_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}-\frac{3P'^1_1(\lambda_{20})\lambda_{10}\omega_2(1,1)k_1u_1}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{10}^2-1)^{\frac{1}{2}}}$$

$$-\frac{3P'^1_1(\lambda_{20})\omega_2(1,2)k_1^2q_1}{Q'^1_1(\lambda_{20})Q'^1_2(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}} \quad \ldots, \quad (35)$$

$$b_2-\sum_{n=1}^{\infty}\phi'_{2n}b_n=-\frac{k_2^2q_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}$$

$$-\frac{5}{3}\,\frac{P'^1_2(\lambda_{20})\lambda_{10}\omega_2(2,1)k_1u_1}{Q'^1_2(\lambda_{20})(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}-\frac{5}{3}\,\frac{P'^1_2(\lambda_{20})\omega_2(2,2)k_1^2q_1}{Q'^1_2(\lambda_{10})Q'^1_2(\lambda_{20})(\lambda_{10}^2-1)^{\frac{1}{2}}} \quad (36)$$

$$b_p - \sum_{n=1}^{\infty} \phi'_{pn} b_n = -\frac{2(2p+1)}{p(p+1)} \frac{P'^1_p(\lambda_{20})}{Q'^1_p(\lambda_{20})} \left[\frac{\lambda_{10}\omega_2(p,1)k_1u_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})} + \frac{\omega_2(p,2)k_1^2q_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{10})} \right] \quad \ldots \quad (37)$$

$$(p=3,4,\ldots ad\ inf.)$$

where $\phi'_{pn} = \frac{n(n+1)(2p+1)}{p(p+1)} \frac{P'^1_p(\lambda_{20})}{Q'^1_p(\lambda_{20})}$

$$\sum_{m=1}^{\infty} (2m+1)\omega_2(p,m)\omega_1(m,n)\frac{P'^1_m(\lambda_{10})}{Q'^1_m(\lambda_{10})} \quad \ldots \quad (38)$$

Proceeding in a similar way, we obtain the equations giving C's and c's. Thus,

$$C_1 - \sum_{n=1}^{\infty} \phi_{1n}C_n = -\frac{\lambda_{10}k_1v_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})} - \frac{3P'^1_1(\lambda_{10})\lambda_{20}\omega_1(1,1)k_2v_2}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}} + \frac{3P'^1_1(\lambda_{10})\omega_1(1,2)k_2^2p_2}{Q'^1_1(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})} \quad \ldots \quad (39)$$

$$C_2 - \sum_{n=1}^{\infty} \phi_{2n}C_n = \frac{k_1^2p_1}{3(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{10})} - \frac{5}{3}\frac{P'^1_2(\lambda_{10})\lambda_{20}\omega_1(2,1)k_2v_2}{Q'^1_2(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}} + \frac{5}{3}\frac{P'^1_2(\lambda_{10})\omega_1(2,2)k_2^2p_2}{Q'^1_2(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})} \quad (40)$$

$$C_p - \sum_{n=1}^{\infty} \phi_{pn}C_n = -\frac{2(2p+1)}{p(p+1)} \frac{P'^1_p(\lambda_{10})}{Q'^1_p(\lambda_{10})} \times \left[\frac{\lambda_{20}\omega_1(p,1)k_2v_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})} - \frac{\omega_1(p,2)k_2^2p_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})} \right] \ldots \quad (41)$$

$$(p=3,4,\ldots ad\ inf.)$$

Also,

$$c_1 - \sum_{n=1}^{\infty} \phi'_{1n}c_n = -\frac{\lambda_{20}k_2v_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})} - \frac{3P'^1_1(\lambda_{20})\lambda_{10}\omega_2(1,1)k_1v_1}{Q'^1_1(\lambda_{20})Q'^1_1(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}} + \frac{3P'^1_1(\lambda_{20})\omega_2(1,2)k_1^2p_1}{Q'^1_1(\lambda_{20})(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{10})} \quad \ldots \quad (42)$$

$$c_2-\sum_{n=1}^{\infty}\phi'_{2n}c_n=\frac{k_2^2p_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}-\frac{5}{3}\,\frac{P'^1_2(\lambda_{20})\lambda_{10}\omega_2(2,1)k_1v_1}{Q'^1_2(\lambda_{20})Q'^1_1(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}$$

$$+\frac{5}{3}\,\frac{P'^1_2(\lambda_{20})\omega_2(2,2)k_1^2p_1}{Q'^1_2(\lambda_{20})Q'^1_2(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\quad\ldots\quad(43)$$

$$c_p-\sum_{n=1}^{\infty}\phi'_{pn}c_n=-\frac{2(2p+1)}{p(p+1)}\,\frac{P'^1_p(\lambda_{20})}{Q'^1_p(\lambda_{20})}\times\left[\frac{\lambda_{10}\omega_2(p,1)k_1v_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}\right.$$

$$\left.-\frac{\omega_2(p,2)k_1^2p_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{10})}\right]\quad\ldots\quad(44)$$

$$(p=3,4\ldots ad\ inf.)$$

where ϕ_{pn}, ϕ'_{pn} are defined as in (34) and (39).

Each of the above sets of equations giving A's, a's, B's, b's, C's, c's may be solved determinantally and it is plain that A's and a's are linear in w_1 and w_2 B's and b's are linear in u_1, u_2, q_1 and q_2, and C's and c's are linear in v_1, v_2 p_1 and p_2. The theory of solutions of such equations has been worked out by Hill, Poincare, Von. Koch, Teoplitz, Hilbert and others. The constants can, therefore, be determined and the problem solved.

4. The complete algebraic values of A's and B's and other constants thus determined are not much suitable for numerical calculations. But the constants may be calculated to any degree of approximation as follows :—

From (7)

$$\omega_1(m,n)=(-)^n\frac{2^n\lfloor n}{\lfloor 2n+1}\ \rho_1^{n+1}\left[D^n_{t_1}+\frac{\rho_1^2}{2\cdot(2n+3)}D^{n+2}_{t_1}+\text{etc.}\right]Q_m(t_1)$$

where $\qquad \rho_1=\dfrac{k_2}{k_1},\quad t_1=\dfrac{s}{k_1}$

$$=\frac{2^{m+n}\lfloor m\lfloor n\lfloor m+n}{\lfloor 2m+1\lfloor 2n+1}\ \frac{k_1^m k_2^{n+1}}{s^{m+n+1}}\left[1+\frac{\lfloor m+n+2}{2\lfloor m+n}\,\frac{1}{s^2}\left(\frac{k_1^2}{2m+3}+\frac{k_2^2}{2m+3}\right)\right.$$

$$+\frac{\lfloor m+n+4}{2\cdot4\cdot\lfloor m+n}\,\frac{1}{s^4}\left\{\frac{k_1^4}{(2m+3)(2m+5)}+\frac{2k_1^2k_2^2}{(2m+3)(2n+3)}\right.$$

$$\left.\left.+\frac{k_2^4}{(2n+3)(2n+5)}\right\}+\text{etc.}\right]\quad\ldots\quad(45)$$

substituting the values of $Q_m(t_1)$ and simplifying.

Thus the lowest order of $\omega_1\ (m,\ n)$ is $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^{m+n+1}$ and such is the case in $\omega_2\ (m, n)$. Hence, from (27), the lowest value of

$$\theta_{p,n}=3\ (2p+1)\ \frac{P'_p\ (\lambda_{10})\ \omega_1\ (p,1)\ \omega_2\ (1,n)}{Q'_p\ (\lambda_{10})\ Q'_1\ (\lambda_{20})}\ \text{taking } m=1$$

$$=\frac{2^{p+n}\ \underline{|p}\ \underline{|n}\ \underline{|p+1}\ \ \underline{|n+1}}{3\ \underline{|2p}\ \ \underline{|2n+1}}\ \ \frac{P'_p\ (\lambda_{10})\ k_1^{\ p+n+1}\ k_2^{\ 3}}{Q'_p\ (\lambda_{10})\ Q'_1\ (\lambda_{20})\ s^{p+n+4}}\ \ldots\ (46)$$

The lowest value of θ'_{pn} is obtained from (46) by interchanging k_1 and k_2, λ_{10} and λ_{20}.

From (34), the lowest value of ϕ_{pn}

$$=\frac{3\ (2p+1)\ n\ (n+1)\ P'_p{}^1\ (\lambda_{10})}{p\ (p+1)\ Q'_p{}^1\ (\lambda_{10})}\ \omega_1\ (p,1)\ \omega_2\ (1,n)\ \frac{P'_1{}^1\ (\lambda_{20})}{Q'_1{}^1\ (\lambda_{20})}$$

$$=\frac{2^{n+p}\ n\ (n+1)\ \ \underline{|p-1}\ \ \underline{|p}\ \underline{|n}\ \underline{|n+1}}{3\ \underline{|2p}\ \ \underline{|2n+1}}\ \ \frac{P'_p{}^1\ (\lambda_{10})\ P'_1{}^1\ (\lambda_{20})}{Q'_p{}^1\ (\lambda_{10})\ Q'_1{}^1\ (\lambda_{20})}$$

$$\frac{k_1^{\ n+p+1}\ k_2^{\ 3}}{s^{n+p+4}}\ \ldots\ (47)$$

The lowest value of ϕ'_{pn} is obtained from (47) by writing k_1 for k_2 and *vice versa*, and interchanging λ_{10} and λ_{20}.

5. (A). If the spheroids are so separated that we can neglect the terms of the order $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^3$ and higher powers, we have, from (25) to (29),

$$A_1=-\ \frac{k_1\ w_1}{Q'_1\ (\lambda_{10})}\ ,\ a_1=-\ \frac{k_2\cdot w_2}{Q'_1\ (\lambda_{20})}$$

$$A_p\ \text{etc.}=0,\ a_p=0;\ \ (p=2,3\ldots ad.\ inf.)$$

From (31) to (37),

$$B_1=-\ \frac{\lambda_{10}\ k_1\ u_1}{(\lambda_{10}{}^2-1)^{\frac{1}{2}}\ Q'_1{}^1\ (\lambda_{10})}\ ;\ B_2=-\ \frac{k_1{}^2\ q_1}{3\ (\lambda_{10}{}^2-1)^{\frac{1}{2}}\ Q'_2{}^1\ (\lambda_{10})}$$

$$B_p=0,\ \ \ \ (p=3,4\ldots ad.\ inf.)$$

$$b_1=-\frac{\lambda_{20}\,k_2\,u_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}\,Q'^1_1\,(\lambda_{20})}\;,\; b_2=-\frac{k_2^2\,q_2}{3\,(\lambda_{20}^2-1)^{\frac{1}{2}}\,Q'^1_2\,(\lambda_{20})}$$

$b_p=0, \qquad (p=3,4\ldots ad.\ inf.)$

From (39) to (44),

$$C_1=-\frac{\lambda_{10}\,k_1\,v_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}\,Q'^1_1\,(\lambda_{10})}\;,\; C_2=\frac{k_1^2\,p_1}{3\,(\lambda_{10}^2-1)^{\frac{1}{2}}\,Q'^1_2\,(\lambda_{10})}$$

$C_p=0 \qquad (p=3,4\ldots ad.\ inf.)$

$$c_1=-\frac{\lambda_{20}\,k_2\,v_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}\,Q'^1_1\,(\lambda_{20})}\;;\; c_2=\frac{k_2^2\,p_2}{3\,(\lambda_{20}^2-1)^{\frac{1}{2}}\,Q'^1_2\,(\lambda_{20})}$$

$c_p=0 \qquad (p=3,4\ldots ad.\ inf.)$

Hence, from (5) $\phi=A_1\,P_1\,(\mu_1)\,Q_1\,(\lambda_1)+a_1\,P_1\,(\mu_2)\,Q_1\,(\lambda_2)$

$$+P_1^1\,(\mu_1)\,Q_1^1\,(\lambda_1)\,(B_1\cos\omega+C_1\sin\omega)$$
$$+P_2^1\,(\mu_1)\,Q_2^1\,(\lambda_1)\,(B_2\cos\omega+C_2\sin\omega)$$
$$+P_1^1\,(\mu_2)\,Q_1^1\,(\lambda_2)\,(b_1\cos\omega+c_1\sin\omega)$$
$$+P_2^1\,(\mu_2)\,Q_2^1\,(\lambda_2)\,(b_2\cos\omega+c_2\sin\omega) \qquad \ldots \quad (48)$$

where A_1, a_1; B_1, B_2, b_1. b_2; C_1, C_2, c_1, c_2 are given above.

(B) If we neglect terms of the order $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^5$ and higher powers, then we have from (25)

$$A_1=-\frac{k_1\,w_1}{Q'_1\,(\lambda_{10})}+\frac{2k_1\,k_2^3\,w_2}{3\,Q'_1(\lambda_{10})\,Q'_1(\lambda_{20})}\,\frac{1}{s^3}\left[\because \omega_1\,(1,1)=\tfrac{2}{3}\frac{k_1\,k_2^2}{s^3}\right]$$

From (28) or from symmetry,

$$a_1=-\frac{k_2\,w_2}{Q'_1\,(\lambda_{20})}+\tfrac{2}{3}\,\frac{k_2\,k_1^3\,w_1}{Q'_1\,(\lambda_{10})\,Q'_2\,(\lambda_{20})}\cdot\frac{1}{s^3}$$

From (26) $A_2=5\,\dfrac{P'_2\,(\lambda_{10})\,\omega_1\,(2,1)}{Q'_2\,(\lambda_{10})\,Q'_1\,(\lambda_{20})}\,k_2\,w_2$

$$=\tfrac{2}{3}\,\frac{P'_2\,(\lambda_{10})\,k_1^2\,k_2^3\,w_2}{Q'_2\,(\lambda_{10})\,Q'_1\,(\lambda_{20})\,s^4}\left[\because \omega_1\,(2,1)=\tfrac{2}{15}\,\frac{k_1^2\,k_2^2}{s^4}\right]$$

$A_p=0, \qquad (p=3,4,\ldots ad.\ inf.)$

From (29),

$$a_2=\tfrac{2}{5}\,\frac{P'_2(\lambda_{20})\,w_1}{Q'_2(\lambda_{20})\,Q'_1(\lambda_{10})}\;\frac{k_1^3 k_2^2}{s^4}$$

$$a_p=0 \qquad (p=3,4,\ldots ad.\ inf.)$$

Also from (31), (32), (33),

$$B_1=-\frac{\lambda_{10}\,k_1\,u_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}-\tfrac{2}{3}\,\frac{P'^1_1(\lambda_{10})\,\lambda_{20}\,k_2\,u_2\,k_1\,k^2_2}{Q'^1_1(\lambda_{10})\,Q'^1_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}s^3}$$

$$-\tfrac{2}{5}\,\frac{P'^1_1(\lambda_{10})\,q_2}{Q'^1_1(\lambda_{10})\,(\lambda_{20}^2-1)^{\frac{1}{2}}\,Q'^1_2(\lambda_{20})}\;\frac{k_1\,k_2^5}{s^4}$$

$$B_2=-\frac{k_1^2\,q_1}{3\,(\lambda_{10}^2-1)^{\frac{1}{2}}\,Q'^1_2(\lambda_{10})}$$

$$-\tfrac{2}{9}\,\frac{P'^1_2(\lambda_{10})\,\lambda_{20}\,k_2\,u_2}{Q'^1_2(\lambda_{10})\,(\lambda_{20}^2-1)^{\frac{1}{2}}\,Q'^1_1(\lambda_{20})}\cdot\frac{k_1^2\,k_2^2}{s^4}$$

$$B_p=0, \qquad (p=3,4,\ldots ad.\ inf.)$$

Similarly, $$b_1=-\frac{\lambda_{20}\,k_2\,u_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}\,Q'^1_1(\lambda_{20})}$$

$$-\tfrac{2}{3}\,\frac{P'^1_1(\lambda_{20})\,\lambda_{10}\,k_1\,u_1}{Q'^1_1(\lambda_{20})\,Q'^1_1(\lambda_{10})\,(\lambda_{10}^2-1)^{\frac{1}{2}}}\cdot\frac{k_2\,k_1^2}{s^3}$$

$$-\tfrac{2}{5}\,\frac{P'^1_1(\lambda_{20})\,q_1\,k_1\,k_2}{Q'^1_1(\lambda_{20})\,(\lambda_{10}^2-1)^{\frac{1}{2}}\,Q'^1_2(\lambda_{10})}\;\frac{k_1^4}{s_4},$$

$$b_2=-\frac{k_2^2q_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}-\frac{2}{9}\,\frac{P'^1_2(\lambda_{20})\lambda_{10}k_1u_1}{Q'^1_2(\lambda_{20})(\lambda_{10}^2-1)^{\frac{1}{2}}Q'(\lambda_{10})}\;\frac{k_1^2k_2^2}{s^4}$$

$$b_p=0, \qquad (p=3,4,\ldots ad\ inf.)$$

From (39), (40), (41), we have,

$$C_1=-\frac{\lambda_{10}k_1v_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}-\frac{2}{3}\,\frac{P'^1_1(\lambda_{10})\lambda_{20}k_2v_2}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}\;\frac{k_1k_2^2}{s^3}$$

$$+\frac{2}{5}\,\frac{P'^1_1(\lambda_{10})p_2k_1k_2}{Q'^1_1(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}\;\frac{k_2^4}{s^4}$$

$$C_2=\frac{k_1^2p_1}{3(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{10})}-\frac{2}{9}\frac{P'^1_2(\lambda_{10})\lambda_{20}k_2v_2}{Q'^1_2(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}\frac{k_1^2k_2^2}{s^4}$$

$$C_p=0,\qquad (p=3,4,5\ldots ad\ inf.)$$

Also,

$$c_1=-\frac{\lambda_{20}k_2v_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}-\frac{2}{3}\frac{P'^1_1(\lambda_{20})\lambda_{10}k_1v_1}{Q'^1_1(\lambda_{20})Q'^1_1(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\frac{k_2k_1^2}{s^3}$$

$$+\frac{2}{5}\frac{P'^1_1(\lambda_{20})k_1k_2p_1}{Q'^1_1(\lambda_{20})Q'^1_2(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}}\frac{k_1^4}{s^4}$$

$$c_2=\frac{k_2^2p_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}-\frac{2}{9}\frac{P'^1_2(\lambda_{20})\lambda_{10}k_1v_1}{Q'^1_2(\lambda_{20})Q'^1_1(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\frac{k_1^2k_2^2}{s^4}$$

$$c_p=0,\qquad (p=3,4,5\ldots ad\ inf.)$$

Hence from (5),

$$\phi=\sum_{n=1}^{\infty}\{A_nP_n(\mu_1)Q_n(\lambda_1)+a_nP_n(\mu_2)Q_n(\lambda_2)$$

$$+P^1_n(\mu_1)Q^1_n(\lambda_1)(B_n\cos\omega+C_n\sin\omega)$$

$$+P^1_n(\mu_2)Q^1_n(\lambda_2)(b_n\cos\omega+c_n\sin\omega)\}\quad\ldots\quad(49)$$

where the constants are given above.

(C) If we neglect terms of the order $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^7$ and highers, powers we have from (25) and (45),

$$A_1=-\frac{k_1w_1}{Q'_1(\lambda_{10})}\left\{1+\theta_{11}\right\}+\frac{3\omega_1(1,1)k_2w_2}{Q'_1(\lambda_{10})Q'_1(\lambda_{20})}$$

$$=-\frac{k_1w_1}{Q'_1(\lambda_{10})}\left\{1+\frac{4}{9}\frac{1}{Q'_1(\lambda_{10})Q'_1(\lambda_{20}}\frac{k_1^3k_2^3}{s^6}\right\}.$$

From (26),

$$A_2=\frac{2}{3}\frac{P'_2(\lambda_{10})k_2w_2}{Q'_2(\lambda_{10})Q'_1(\lambda_{20})}\frac{k_1^2k_2^2}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{7}+\frac{k_2^2}{5}\right)\right\}$$

$$A_3=7\frac{P'_3(\lambda_{10})}{Q'_3(\lambda_{10})}\frac{\omega_1(3,1)}{Q'_1(\lambda_{20})}k_2w_2=\frac{8}{15}\frac{P'_3(\lambda_{10})k_2w_2}{Q_3(\lambda_{10})Q'_1(\lambda_{20})}\frac{k_1^3k_2^2}{s^5}$$

$$\left[\because\omega_1(3,1)=\frac{8}{105}\frac{k_1^3k_2^2}{s^5}\right]$$

$$A_4 = 9\frac{P'_4(\lambda_{10})\omega_1(4,1)}{Q'_4(\lambda_{10})Q'_1(\lambda_{20})}k_2 w_2$$

$$= \frac{8}{21}\frac{P'_4(\lambda_{10})k_2 w_2}{Q'_4(\lambda_{10})Q'_1(\lambda_{20})}\frac{k_1^4 k_2^2}{s^6}\left[\because \omega_1(4,1) = \frac{8}{189}\frac{k_1^4 k_2^2}{s^6}\right]$$

$$A_p = 0, \qquad (p = 5, 6 \ldots ad\ inf.)$$

Similarly,

$$a_1 = -\frac{k_2 w_2}{Q'_1(\lambda_{20})}\left\{1 + \frac{4}{9}\cdot\frac{1}{Q'_1(\lambda_{10})Q'(\lambda_{20})}\frac{k_1^3 k_2^3}{s^6}\right\}$$

$$+\frac{2}{3}\frac{k_1 w_1}{Q'_1(\lambda_{10})Q'_1(\lambda_{20})}\frac{k_2 k_1^2}{s^3}\left\{1 + \frac{6}{5}\frac{k_1^2 + k_2^2}{s^2}\right\}$$

$$a_2 = \frac{2}{3}\frac{P'_2(\lambda_{20})k_1 w_1}{Q'_2(\lambda_{20})Q'_1(\lambda_{10})}\frac{k_1^2 k_2^2}{s^4}\left\{1 + \frac{10}{s^2}\left(\frac{k_2^2}{7} + \frac{k_1^2}{5}\right)\right\}$$

$$a_3 = \frac{8}{15}\frac{P'_3(\lambda_{20})k_1 w_1}{Q'_3(\lambda_{20})Q'_1(\lambda_{10})}\frac{k_2^3 k_1^2}{s^5}$$

$$a_4 = \frac{8}{21}\frac{P'(\lambda_{20})k_1 w_1}{Q'_4(\lambda_{20})Q'_1(\lambda_{10})}\frac{k_1^2 k_2^4}{s^6}$$

$$a_p = 0, \qquad (p = 5, 6, \ldots ad\ inf.)$$

From (31),

$$B_1 = -\frac{\lambda_{10}k_1 u_1}{(\lambda_{10}^2 - 1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}\left(1 + \phi_{11}\right)$$

$$-\frac{2}{3}\frac{P'^1_1(\lambda_{10})\lambda_{20}k_2 u_2}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{20}^2 - 1)^{\frac{1}{2}}}\frac{k_1 k_2^2}{s^3}\left(1 + \frac{6}{5}\frac{k_1^2 + k_2^2}{s^2}\right)$$

$$-\frac{2}{5}\frac{P'^1_1(\lambda_{10})k_1 k_2 q_2}{Q'^1_1(\lambda_{10})Q'^1_2(\lambda_{20})(\lambda_{20}^2 - 1)^{\frac{1}{2}}}\frac{k_2^4}{s^4}\left\{1 + \frac{10}{s^2}\left(\frac{k_1^2}{5} + \frac{k_2^2}{7}\right)\right\}$$

$$= -\frac{\lambda_{10}k_1 u_1}{(\lambda_{10}^2 - 1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}\left(1 + \frac{4}{9}\frac{P'^1_1(\lambda_{10})P'^1_1(\lambda_{20})}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})}\frac{k_1^3 k_2^3}{s^6}\right)$$

$$-\frac{2}{3}\frac{P'^1_1(\lambda_{10})\lambda_{20}k_2 u_2}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{20}^2 - 1)^{\frac{1}{2}}}\frac{k_1 k_2^2}{s^3}\left(1 + \frac{6}{5}\frac{k_1^2 + k_2^2}{s^2}\right)$$

$$-\frac{2}{5}\frac{P'^1_1(_{10})k_1 k_2 q_2}{Q'^1_1(\lambda_{10})Q'^1_2(\lambda_{20})(\lambda_{20}^2 - 1)^{\frac{1}{2}}}\frac{k_2^4}{s_2^4}\left\{1 + \frac{10}{s^2}\left(\frac{k_1^2}{5} + \frac{k_2^2}{7}\right)\right\}$$

$$B_2=-\frac{k_1^2q_1}{3(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{10})}$$

$$-\frac{2}{9}\frac{P'^1_2(\lambda_{10})\lambda_{20}k_2u_2}{Q'^1_2(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}\frac{k_1^2k_2^2}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{7}+\frac{k_2^2}{5}\right)\right\}$$

$$-\frac{8}{45}\frac{P'^1_2(\lambda_{10})k_2^2q_2}{Q'^1_2(\lambda_{10})Q'^1_2(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}\frac{k_1^2k_2^3}{s^5}$$

$$B_3=-\frac{4}{45}\frac{P'^1_3(\lambda_{10})}{Q'^1_3(\lambda_{10})}\Bigg[\frac{\lambda_{20}k_2u_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}\frac{k_1^3k_2^2}{s^5}$$

$$+\frac{q_2k_2^2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}\frac{k_1^3k_2^3}{s^6}\Bigg]$$

$$B_4=-\frac{4}{105}\frac{P'^1_4(\lambda_{10})\lambda_{20}k_2u_2}{Q'^1_4(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}\frac{k_1^4k_2^2}{s^6}$$

$$B_p=0,\qquad (p=5,6\ldots ad\ inf.)$$

From (35) to (38), we have in a similar way,

$$b_1=-\frac{\lambda_{20}k_2u_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}\left(1+\frac{4}{9}\frac{P'^1_1(\lambda_{10})P'^1_1(\lambda_{20})}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})}\frac{k_1^3k_2^3}{s^6}\right)$$

$$-\frac{2}{3}\frac{P'^1_1(\lambda_{20})\lambda_{10}k_1u_1}{Q'^1_1(\lambda_{20})Q'^1_1(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\frac{k_2k_1^2}{s^3}\left(1+\frac{6}{5}\frac{k_1^2+k_2^2}{s^2}\right)$$

$$-\frac{2}{5}\frac{P'^1_1(\lambda_{20})k_1k_2q_1}{Q'^1_1(\lambda_{20})Q'^1_2(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\frac{k_1^4}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{7}+\frac{k_2^2}{5}\right)\right\}$$

$$b_2=-\frac{k_2^2q_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}-\frac{2}{9}\frac{P'^1_2(\lambda_{20})\lambda_{10}k_1u_1}{Q'^1_2(\lambda_{20})Q'^1_1(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\frac{k_1^2k_2^2}{s^4}$$

$$\times\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{5}+\frac{k_2^2}{7}\right)\right\}-\frac{8}{45}\frac{P'^1_2(\lambda_{20})k_1^2q_1}{Q'^1_2(\lambda_{20})Q'^1_2(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\frac{k_1^3k_2^2}{s^5}$$

$$b_3=-\frac{4}{45}\frac{P'^1_3(\lambda_{20})}{Q'^1_3(\lambda_{20})}\Bigg[\frac{\lambda_{10}k_1u_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}\frac{k_1^2k_2^3}{s^5}$$

$$+\frac{q_1k_1^2}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{10})}\frac{k_1^3k_2^3}{s^6}\Bigg]$$

$$b_4=-\frac{4}{105}\frac{P'^1_4(\lambda_{20})}{Q'^1_4(\lambda_{20})}\frac{\lambda_{10}k_1u_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}\frac{k_1^2k_2^4}{s^6}$$

$$b_p=0,\qquad (p=5,6,\ldots ad\ inf.)$$

From (39) to (44), we have,

$$C_1=-\frac{\lambda_{10}k_1v_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q_1^{\prime 1}(\lambda_{10})}\left(1+\frac{4}{9}\,\frac{P_1^{\prime 1}(\lambda_{10})P_1^{\prime 1}(\lambda_{20})}{Q_1^{\prime 1}(\lambda_{10})Q_1^{\prime 1}(\lambda_{20})}\,\frac{k_1^3k_2^3}{s^6}\right)$$

$$-\frac{2}{3}\,\frac{P_1^{\prime 1}(\lambda_{10})\lambda_{20}k_2v_2}{Q_1^{\prime 1}(\lambda_{10})Q_1^{\prime 1}(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}\,\frac{k_1k_2^3}{s^3}\left(1+\frac{6}{5}\,\frac{k_1^2k_2^2}{s^2}\right)$$

$$+\frac{2}{5}\,\frac{P_1^{\prime 1}(\lambda_{10})k_1k_2p_2}{Q_1^{\prime 1}(\lambda_{10})Q_2^{\prime 1}(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}\,\frac{k_1^2k_2^4}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{5}+\frac{k_2^2}{7}\right)\right\}$$

$$C_2=\frac{k_1^2p_1}{3(\lambda_{10}^2-1)^{\frac{1}{2}}Q_2^{\prime 1}(\lambda_{10})}-\frac{2}{9}\,\frac{P_2^{\prime 1}(\lambda_{10})\lambda_{20}k_2v_2}{Q_2^{\prime 1}(\lambda_{10})Q_1^{\prime 1}(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}$$

$$\times\frac{k_1^2k_2^2}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{7}+\frac{k_2^2}{5}\right)\right\}$$

$$+\frac{8}{45}\,\frac{P_2^{\prime 1}(\lambda_{10})k_2^2p_2}{Q_2^{\prime 1}(\lambda_{10})Q_2^{\prime 1}(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}\,\frac{k_1^3k_2^3}{s^5}$$

$$C_3=-\frac{4}{45}\,\frac{P_3^{\prime 1}(\lambda_{10})}{Q_3^{\prime 1}(\lambda_{10})}\left[\frac{\lambda_{20}k_2v_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q_1^{\prime 1}(\lambda_{20})}\,\frac{k_1^3k_2^2}{s^5}\right.$$

$$\left.-\frac{p_2k_2^2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q_2^{\prime 1}(\lambda_{20})}\,\frac{k_1^3k_2^3}{s^6}\right]$$

$$C_4=-\frac{4}{105}\,\frac{P_4^{\prime 1}(\lambda_{10})\lambda_{20}k_2v_2}{Q_4^{\prime 1}(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q_1^{\prime 1}(\lambda_{20})}\,\frac{k_1^4k_2^2}{s^6}$$

$$C_p=0, \qquad (p=5,6,\ldots\textit{ad inf.})$$

$$c_1=-\frac{\lambda_{20}k_2v_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q_1^{\prime 1}(\lambda_{20})}\left(1+\frac{4}{9}\,\frac{P_1^{\prime 1}(\lambda_{10})P_1^{\prime 1}(\lambda_{20})}{Q_1^{\prime 1}(\lambda_{10})Q_1^{\prime 1}(\lambda_{20})}\,\frac{k_1^3k_2^3}{s^6}\right)$$

$$-\frac{2}{3}\,\frac{P_1^{\prime 1}(\lambda_{20})\lambda_{10}k_1v_1}{Q_1^{\prime 1}(\lambda_{20})Q_1^{\prime 1}(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\,\frac{k_2k_1^3}{s^3}\left(1+\frac{6}{5}\,\frac{k_1^2+k_2^2}{s^2}\right)$$

$$+\frac{2}{5}\,\frac{P_1^{\prime 1}(\lambda_{20})k_1k_2p_1}{Q_1^{\prime 1}(\lambda_{20})Q_2^{\prime 1}(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\,\frac{k_1^4}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{7}+\frac{k_2^2}{5}\right)\right\}$$

$$c_2=\frac{k_2^2p_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}}Q_2^{\prime 1}(\lambda_{20})}$$

$$-\frac{2}{9}\,\frac{P_2^{\prime 1}(\lambda_{20})\lambda_{10}k_1v_1}{Q_2^{\prime 1}(\lambda_{20})Q_1^{\prime 1}(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\,\frac{k_1^2k_2^2}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{5}+\frac{k_2^2}{7}\right)\right\}$$

$$+\frac{8}{45}\,\frac{P_2^{\prime 1}(\lambda_{20})k_1^2p_1}{Q_2^{\prime 1}(\lambda_{20})Q_2^{\prime 1}(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\,\frac{k_1^3k_2^3}{s^6}$$

$$c_3=-\frac{4}{45}\,\frac{P'^1_3(\lambda_{20})}{Q'^1_3(\lambda_{20})}\left[\frac{\lambda_{10}k_1v_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}\;\frac{k_1^2k_2^3}{s^5}\right.$$

$$\left.-\frac{p_1k_1^2}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{10})}\;\frac{k_1^3k_2^3}{s^6}\right]$$

$$c_4=-\frac{4}{105}\,\frac{P'^1_4(\lambda_{20})}{Q'^1_4(\lambda_{20})}\,\frac{\lambda_{10}k_1v_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}\;\frac{k_1^2k_2^4}{s^6}$$

$$c_p=0, \qquad (p=5,6,\dots ad\ inf.)$$

$\therefore$ From (5),

$$\phi=\sum_{n=1}^{4}\{A_nP_n(\mu_1)Q_n(\lambda_1)+a_nP_n(\mu_2)Q_n(\lambda_2)$$

$$+P_n^1(\mu_1)Q_n^1(\lambda_1)(B_n\cos\omega+C_n\sin\omega)$$

$$+P_n^1(\mu_2)Q_n^1(\lambda_2)(b_n\cos\omega+c_n\sin\omega)\}$$

where the constants are determined above. ... (50)

In a similar way the expression for ϕ may be determined correct to any power of $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)$.

6. ϕ *in spherical harmonics.*

ϕ may also be expressed in spherical harmonics by the help of the following theorem :

$$P_n^\sigma(\mu)Q_n^\sigma(\lambda)=\mp(-)^\sigma\,\frac{2^n\lfloor n\,\lfloor n+\sigma}{\lfloor 2n+1}\;k^{n+1}\left[\frac{P_n^\sigma(\cos\theta)}{r^{n+1}}\right.$$

$$\left.+\frac{\lfloor n+2-\sigma}{\lfloor n-\sigma}\;\frac{k^2}{2(2n+3)}\;\frac{P_{n+2}^\sigma(\cos\theta)}{r^{n+3}}+\text{etc.}\right]$$

for all integral values of σ including zero. ... (51)

Also, $\dfrac{1}{\lambda_{10}}=e_1,\ \dfrac{1}{\lambda_{20}}=e_2,\ k_1=a_1e_1,\ k_2=a_2e_2.$

7. *Motion of two oblate spheroids.*

If in Art. 2, we write $\frac{k'_1}{i}$ for k and $i\lambda$ for λ,

we have,
$$x_1 = k'_1(1-\mu_1^2)^{\frac{1}{2}}(\lambda_1^2+1)^{\frac{1}{2}}\cos\omega$$
$$y_1 = k'_1(1-\mu_1^2)^{\frac{1}{2}}(\lambda_1^2+1)^{\frac{1}{2}}\sin\omega$$
$$z_1 = k'_1\mu_1\lambda_1$$
$$x_2 = k'_2(1-\mu_2^2)^{\frac{1}{2}}(\lambda_2^2+1)^{\frac{1}{2}}\cos\omega$$
$$y_2 = k'_2(1-\mu_2^2)^{\frac{1}{2}}(\lambda_2^2+1)^{\frac{1}{2}}\sin\omega$$
$$z_2 = k'_2\mu_2\lambda_2$$

where $(\lambda_1, \mu_1, \omega)$, $(\lambda_2, \mu_2, \omega)$ are two systems of planetary spheroidal co-ordinates so that $\lambda_1 = \lambda_{10}$ and $\lambda_2 = \lambda_{20}$ on the given spheroids. Hence, by writing $\frac{k'}{i}$ for k and $i\lambda$ for λ in (5) and the expressions for the constants in the case of prolate spheroids, the corresponding results for oblate spheroids are obtained at once.

8. If $u_1, v_1, u_2, v_2, p_1, q_1, p_2, q_2$ are all zero, evidently the problem reduces to that of the motion[1] of two spheroids in an infinite liquid along their common axis of revolution. In this case, we have from 5(A),

$$A_1 = -\frac{k_1 w_1}{Q'_1(\lambda_{10})}, \quad a_1 = -\frac{k_2 w_2}{Q'_1(\lambda_{20})} \text{ and all B's, } b\text{'s, C's, } c\text{'s zero}$$

and from 5(B),

$$A_1 = -\frac{k_1 w_1}{Q'(\lambda_{10})} + \frac{2}{3}\ \frac{k_2 w_2}{Q'_1(\lambda_{10})Q'(\lambda_{20})}\ \frac{k_1 k_2^2}{s^3}$$

$$a_1 = -\frac{-k_2 w_2}{Q'_1(\lambda_{20})} + \frac{2}{3}\ \frac{k_1 w_1}{Q'_1(\lambda_{10})Q'(\lambda_{20})}\ \frac{k_2 k_1^2}{s^3}.$$

All B's, C's, b's, c's are zero.

Both these results have already been obtained by Dr. Datta.[2]

[1] Dr. Bibhutibhushan Datta.—"Americ. Journ. Math." *ibid.*
[2] "Americ. Journ. Math." *ibid*, p. 141.

7

ON THE EVALUATION OF SOME FACTORABLE CONTINUANTS

BY

SATISCHANDRA CHAKRABARTI, M.Sc.

There is a special class of continuants which are resolvable into linear factors, the earliest continuants of this class being those of Sylvester[1] and Painvin.[2] Subsequently Dr. T. Muir[3] and W. H. Metzler[4] took up the subject in right earnest and obtained some continuants of this class. Haripada Datta in his paper, "On the Failure of Heilermann's Theorem"[5] pointed out that "for every finite series we obtain a factorizable continuant." In the same paper of Mr. Datta, a continuant derived from recurring series has been given Here in this paper in article 7, we have evaluated this continuant determinantally. There is another factorable continuant obtained from $\left(\frac{x+1}{x-1}\right)^q$, which is also treated in articles 5 and 6.[6] In evaluating the second continuant we are to apply some algebraic relations which are first established.

[1] Sylvester J. J. "Theoreme sur les determinants de M. Sylvester." Nouv Annals de Math., xiii, p. 305 ; or The Theory of Determinants in the Historical Order of Development by Muir T., Vol. 2, p. 425.

[2] Painvin L. "Sur un certain systeme d'equations lineaires." Journ, (de Liouville) de Math. (2) iii, pp. 41-46 ; or The Theory of Determinants in the Historical Order of Development by Muir T., Vol. 2, pp. 432-434.

[3] Muir T. "Continuants resolvable into linear factors" Trans. Edin. Roy. Soc. 41, 1905 (343-358). Muir T. "Factorizable continuants" Trans. S. Afric Philos. Soc. 15 pt. 1, 1904 (29-33).

[4] W. H. Metzler. "Some factorable continuants," Edin. Proc. Roy. Soc., 34, 1914 (223-229).

[5] Proc. Edin Math. Soc., Vol. 35 (part 2), session 1916-1917 ; or University Edin. Math. Depart. session 1917, Research paper No 7, p. 12

[6] Haripada Datta "On the Theory of continued fractions." Proc Edin. Math. Soc., Vol. 34 (part 2), session 1915-1916 ; or University Edin. Math. Dept. session 1916, Research paper No. 4, p. 9, Ex. 2.

1. $\{(x+1)(x+3)(x+5)\ldots(x+2r-1)\}$

$$=\{(x-\delta)(x-\delta-2)(x-\delta-4)\ldots(x-\delta-\overline{2r-2})\}$$
$$+(\delta+2r-1)^{r}C_1\{(x-\delta)(x-\delta-2)\ldots(x-\delta-\overline{2r-4})\}$$
$$+(\delta+2r-1)(\delta+2r-3)^{r}C_2\{(x-\delta)(x-\delta-2)\ldots(x-\delta-\overline{2r-6})\}$$
$$+\ldots+\{(\delta+2r-1)(\delta+2r-3)\ldots(\delta+5)(\delta+3)\}^{r}C_{r-1}(x-\delta)$$
$$+\{(\delta+2r-1)(\delta+2r-3)\ldots(\delta+3)(\delta+1)\} \text{ identically} \quad \ldots \quad (1)$$

Let us take the particular case,

$$(x+1)(x+3)(x+5)(x+7)=(x-\delta)(x-\delta-2)(x-\delta-4)(x-\delta-6)$$
$$+(\delta+7)^{4}C_1(x-\delta)(x-\delta-2)(x-\delta-4)$$
$$+(\delta+7)(\delta+5)^{4}C_2(x-\delta)(x-\delta-2)$$
$$+(\delta+7)(\delta+5)(\delta+3)^{4}C_3(x-\delta)+(\delta+7)(\delta+5)(\delta+3)(\delta+1) \ldots \quad (2)$$

and let α and β respectively denote the left-hand-side and the right-hand-side expression of (2).

When $x=\delta$, $\alpha=\beta=(\delta+1)(\delta+3)(\delta+5)(\delta+7)$ and when $x=-1$, $\alpha=0, \beta=(-1)^4(\delta+1)(\delta+3)(\delta+5)(\delta+7)\{1-{}^4C_1+{}^4C_2-{}^4C_3+{}^4C_4\}=0$. When $x=-3$, -5 or -7, we can show, by means of difference formulae that in each of these substitutions $\alpha=\beta=0$. Thus the equation (2) is satisfied for more than four values of x; hence it is an identity. The general case may be similarly proved.

2. $\{(a+2r-1)(a+2r+1)\ldots(a+4r-3)\}$

$$=\{a(a+2)(a+4)\ldots(a+2r-2)\}^{2r}C_{2r}$$
$$+1\cdot{}^{2r}C_{2r-2}\{(a+2)(a+4)\ldots(a+2r-2)\}$$
$$+1\cdot3\cdot{}^{2r}C_{2r-4}\{(a+4)(a+6)\ldots(a+2r-2)\}+\ldots$$
$$+\{1\cdot3\cdot5\ldots(2r-2k-3)\}^{2r}C_{2k+2}\{(a+2r-2k-2)(a+2r-2k)\ldots$$
$$(a+2r-2)\}+\ldots\{1\cdot3\ldots(2r-3)\}^{2r}C_2\ (a+2r+2)$$
$$+\{1\cdot3\cdot5\ldots(2r-1)\}^{2r}C_0 \text{ identically} \quad \ldots \quad (3)$$

Suppose

$$\{(a+2r-1)(a+2r+1)\ldots(a+4r-3)\}=\{a(a+2)\ldots(a+2r-2)\}$$
$$+A_2\{(a+2)(a+4)\ldots(a+2r-2)\}+A_3\{(a+4)(a+6)\ldots(a+2r-2)\}$$
$$+\ldots+A_{r-k-1}\{(a+2r-2k-4)(a+2r-2k-2)\ldots(a+2r-2)\}$$
$$+A_{r-k}\{(a+2r-2k-2)(a+2r-2k)\ldots(a+2r-2)\}+\ldots$$
$$+A_r(a+2r-2)+A_{r+1} \quad \ldots \quad (4)$$

where $A_2, A_3, \ldots A_{r+1}$ which are independent of a, are to be determined.

In (4) put $a=-2r+2$, then $A_{r+1}=\{1\cdot3\cdot5\ldots(2r-1)\}\,{}^{2r}C_0$

$$a=-2r+4, \qquad A_r=\{1\cdot3\cdot5\ldots(2r-3)\}\,{}^{2r}C_2$$

$$a=-2r+6, \qquad A_{r-1}=\{1\cdot3\ldots(2r-5)\}\,{}^{2r}C_4$$

Thus we see that there is a formula for A_{r+1}, A_r and A_{r-1}. Let us assume that this formula is true for all cases from A_{r+1} to A_{r-k}. It is required to show that the formula also holds in the case of A_{r-k-1}.

In (4) putting $a=-2r+2k+6$, we have,

$$\begin{aligned}
\{2\cdot4\cdot6\cdots(2k+4)\}A_{r-k-1} &= \{(2k+5)(2k+7)\ldots(2r+2k+3)\}\\
&\quad-\{4\cdot6\ldots(2k+4)\}A_{r-k}-\{6\cdot8\ldots(2k+4)\}A_{r-k+1}-\cdots\\
&\quad-\{(2k-2m+6)(2k-2m+8)\ldots(2k+4)\}A_{r-m+1}-\ldots\\
&\quad-\{(2k+2)(2k+4)\}A_{r-1}-(2k+4)A_r-A_{r+1}
\end{aligned}$$

$$\begin{aligned}
&=\{(2k+5)(2k+7)\ldots(2r-1)\}\Big[\{(2r+1)(2r+3)\ldots(2r+2k+3)\}\\
&\quad-\frac{\{4\cdot6\ldots(2k+4)\}\{1\cdot3\ldots(2k+3)\}}{\{(2r-1)(2r-3)\ldots(2r-2k-1)\}}\;\frac{|\underline{2r}}{|\underline{2r-2k-2}\;|\underline{2k+2}}\\
&\quad-\frac{\{6\cdot8\ldots(2k+4)\}\{1\cdot3\ldots(2k+3)\}}{\{(2r-1)(2r-3)\ldots(2r-2k+1)\}}\;\frac{|\underline{2r}}{|\underline{2r-2k}\;|\underline{2k}}-\cdots\\
&\quad-\frac{\{(2k-2m+6)(2k-2m+8)\ldots(2k+4)\}\{1\cdot3\ldots(2k+3)\}|\underline{2r}}{\{(2r-1)(2r-3)\ldots(2r-2m+1)\}\;|\underline{2r-2m}\;|\underline{2m}}-\cdots\\
&\quad-\frac{(2k+4)\{1\cdot3\ldots(2k+3)\}}{2r-1}\;\frac{|\underline{2r}}{|\underline{2r-2}\;|\underline{2}}-\{1\cdot3\ldots(2k+3)\}\Big]
\end{aligned}$$

$$\begin{aligned}
&=\{(2k+5)(2k+7)\ldots(2r-1)\}[\{(2r+1)(2r+3)\ldots(2r+2k+3)\}\\
&\quad-(2k+3)\,{}^{k+2}C_1\{2r(2r-2)\ldots(2r-2k)\}\\
&\quad-(2k+3)(2k+1)\,{}^{k+2}C_2\{2r(2r-2)\ldots(2r-2k+2)\}-\ldots\\
&\quad-\{(2k+3)(2k+1)\ldots(2m+1)\}\,{}^{k+2}C_{k-m+2}\{2r(2r-2)\ldots(2r-2m+2)\}\\
&\quad-\ldots-\{(2k+3)(2k+1)\ldots5\cdot3\}\,{}^{k+2}C_{k+1}2r\\
&\quad-\{(2k+3)(2k+1)\ldots3\cdot1\}\,{}^{k+2}C_{k+2}]
\end{aligned}$$

For $\dfrac{\{(2k-2m+6)(2k-2m+8)\ldots(2k+4)\}\{1\cdot3\ldots(2k+3)\}}{\{(2r-1)(2r-3)\ldots(2r-2m+1)\}} \dfrac{|2r}{|2r-2m\,|2m}$

$$=\frac{\{(2k-2m+6)(2k-2m+8)\ldots(2k+4)\}\{1\cdot3\ldots(2k+3)\}\{2r(2r-2)\ldots(2r-2m+2)\}}{|2m}$$

$$=\frac{|2k+4\;\{2r(2r-2)\ldots(2r-2m+2)\}}{\{2\cdot4\cdot6\ldots(2k-2m+4)\}\,|2m}$$

$$=\{(2k+3)(2k+1)\ldots(2m+1)\}\,{}^{k+2}C_{k-m+2}\{2r(2r-2)\ldots(2r-2m+2)\}$$

Hence by the identity obtained from (1) by putting $\delta=0$ we have,

$$\{2\cdot4\cdot6\ldots(2k+4)\}A_{r-k-1}$$

$$=\{(2k+5)(2k+7)\ldots(2r-1)\}\{2r(2r-2)(2r-4)\ldots(2r-2k-2)\}$$

$$=\frac{|2r}{|2k+4\{(2k+6)(2k+8)\cdots(2r-2k-4)\}}$$

$$\therefore\; A_{r-k-1}=\{1\cdot3\cdot5\ldots(2r-2k-5)\}\,{}^{2r}C_{2k+4}.$$

Thus the formula holds universally. Hence the identity is established,

3. $\{(a+2r-1)(a+2r+1)(a+2r+3)\ldots(a+4r-5)\}$

$=\{a(a+2)\ldots(a+2r-4)\}+1\cdot{}^{2r-1}C_{2r-3}\{(a+2)(a+4)\ldots(a+2r-4)\}$

$+1\cdot3\cdot{}^{2r-1}C_{2r-5}\{(a+4)(a+6)\ldots(a+2r-4)\}$

$+1\cdot3\cdot5\cdot{}^{2r-1}C_{2r-7}\{(a+6)(a+8)\ldots(a+2r-4)\}+\cdots$

$+\{1\cdot3\cdot5\ldots(2r-5)\}\,{}^{2r-1}C_3(a+2r-4)$

$+\{1\cdot3\cdot5\ldots(2r-3)\}\,{}^{2r-1}C_1$ identically ... (5)

This theorem is established by proceeding in the same manner as the theorem (3), the identity to be applied being obtained by putting $\delta=2$ in the theorem (1).

4. Here in this article we shall give another identity,

$$\frac{1}{|n}-\frac{1}{|n-1\,|1}+\frac{1}{|n-2\,|2}\;\frac{a+1}{a}-\frac{1}{|n-3\,|3}\;\frac{a+3}{a}$$

$$+\frac{1}{|n-4\,|4}\;\frac{(a+3)(a+5)}{a(a+2)}-\frac{1}{|n-5\,|5}\;\frac{(a+5)(a+7)}{a(a+2)}$$

$$+\frac{1}{|n-6\,|6}\;\frac{(a+5)(a+7)(a+9)}{a(a+2)(a+4)}-\frac{1}{|n-7\,|7}\;\frac{(a+7)(a+9)(a+11)}{a(a+2)(a+4)}+$$

$$\ldots=0 \text{ or } \frac{\{1\cdot3\cdot5\ldots(n-1)\}}{|n\;a(a+2)\ldots(a+n-2)}$$

according as n is odd or even ; the last term of the series is,

$$\frac{\{(a+n)(a+n+2)\dots(a+2n-3)\}}{\lfloor n\{a(a+2)(a+4)\dots(a+n-3)\}}$$

or $$\frac{\{(a+n-1)(a+n+1)\dots(a+2n-3)\}}{\lfloor n\{a(a+2)(a+4)\dots(a+n-2)\}} \quad \dots \quad (6)$$

according as n is odd or even.

Let $n=9$ an odd number, then the series becomes

$$\frac{1}{\lfloor 9} - \frac{1}{\lfloor 8\,\lfloor 1} + \frac{1}{\lfloor 7\,\lfloor 2}\,\frac{a+1}{a} - \frac{1}{\lfloor 6\,\lfloor 3}\,\frac{a+3}{a} + \frac{1}{\lfloor 5\,\lfloor 4}\,\frac{(a+3)(a+5)}{a(a+2)} - \frac{1}{\lfloor 4\,\lfloor 5}\,\frac{(a+5)(a+7)}{a(a+2)}$$

$$+\frac{1}{\lfloor 3\,\lfloor 6}\,\frac{(a+5)(a+7)(a+9)}{a(a+2)(a+4)} - \frac{1}{\lfloor 2\,\lfloor 7}\,\frac{(a+7)(a+9)(a+11)}{a(a+2)(a+4)}$$

$$+\frac{1}{\lfloor 1\,\lfloor 8}\,\frac{(a+7)(a+9)(a+11)(a+13)}{a(a+2)(a+4)(a+6)}$$

$$-\frac{1}{\lfloor 9}\,\frac{(a+9)(a+11)(a+13)(a+15)}{a(a+2)(a+4)(a+6)} \quad \dots \quad (7)$$

Let $${}^n a_r = {}^n C_{r-1} - {}^r C_1\,{}^n C_r + {}^{r+1} C_2\,{}^n C_{r+1} - {}^{r+2} C_3\,{}^n C_{r+2}$$
$$+\dots(-1)^{n-r+1}\;{}^n C_{n-r+1}\,{}^n C_n.$$

Then

$${}^n a_r = {}^n C_{r-1}\{1 - {}^{n-r+1}C_1 + {}^{n-r+1}C_2 - \dots + (-1)^{n-r+1}\;{}^{n-r+1}C_{n-r+1}\} = 0$$

for $${}^q C_k \cdot {}^n C_q = {}^n C_{q-k} \cdot {}^{n-q+k} C_k.$$

Hence $${}^n a_1 = {}^n a_2 = {}^n a_3 = \dots = {}^n a_n = 0 \text{ but } {}^n a_{n+1} = {}^n C_n = 1. \quad \dots \quad (8)$$

Now if $u_1 = a(a+2)(a+4)(a+6)$ which is the denominator of the last term of the series (7),

$$u_2 = (a+2)(a+4)(a+6),\ u_3 = (a+4)(a+6) \text{ and } u_4 = a+6,$$

then the numerator of the series (7) which is

$$a(a+2)(a+4)(a+6) - {}^9C_1 a(a+2)(a+4)(a+6)$$

$$+\,{}^9C_2(a+1)(a+2)(a+4)(a+6) - {}^9C_3(a+3)(a+2)(a+4)(a+6)$$

$$+\,{}^9C_4(a+3)(a+5)(a+4)(a+6) - \dots - {}^9C_9(a+9)(a+11)(a+13)(a+15)$$

may be written as

$$u_1-{}^9C_1u_1+{}^9C_2(u_1+u_2)-{}^9C_3(u_1+{}^3C_1u_2)+{}^9C_4(u_1+{}^4C_2u_2$$
$$+1\cdot3\cdot{}^4C_0u_3)-\ldots+{}^9C_8(u_1+{}^8C_6u_2+1\cdot3\cdot{}^8C_4u_3+1\cdot3\cdot5\cdot{}^8C_2u_4$$
$$+1\cdot3\cdot5\cdot7)-{}^9C_9(u_1+{}^9C_7u_2+1\cdot3\cdot{}^9C_5u_3+1\cdot3\cdot5\cdot{}^9C_3u_4$$
$$+1\cdot3\cdot5\cdot7\cdot{}^9C_1)$$

by applying the theorem (3) to odd terms and the theorem (5) to even terms.

Hence the numerator

$$=u_1{}^9a_1+u_2{}^9a_3+1\cdot3u_3{}^9a_5+1\cdot3\cdot5u_4{}^9a_7+1\cdot3\cdot5\cdot7{}^9a_9=0 \text{ by (8).}$$

Thus the series vanishes.

When $n=6$ an even number and if,

$$u_1=a(a+2)(a+4),\ u_2=(a+2)(a+4) \text{ and } u_3=a+4,$$

the numerator becomes,

$$u_1-{}^6C_1u_1+{}^6C_2(u_1+u_2)-{}^6C_3(u_1+{}^3C_1u_2)+{}^6C_4(u_1+{}^4C_2u_2+1\cdot3u_3)$$
$$-{}^6C_5(u_1+{}^5C_3u_2+1\cdot3\cdot{}^5C_1u_3)+{}^6C_6(u_1+{}^6C_4u_2+1\cdot3\cdot{}^6C_2u_3+1\cdot3\cdot5)$$
$$={}^6a_1u_1+{}^6a_3u_2+1\cdot3{}^6a_5u_3+1\cdot3\cdot5=1\cdot3\cdot5.$$

Hence when $n=6$, the series $=\dfrac{1\cdot3\cdot5}{\underline{|6}\, a(a+2)(a+4)}$.

The general case may be treated exactly in the same manner.

5. We shall now evaluate the factorable continuant

$$D_5=\begin{vmatrix} x-5, & -1 & & & \\ 5^2-1^2 & 3x & -1 & & \\ & 5^2-2^2 & 5x & -1 & \\ & & 5^2-3^2 & 7x & -1 \\ & & & 5^2-4^2 & 9x \end{vmatrix}$$

On this if we proform the operation

$(-1)^{5-1}\,\pi\,(5-1,1)\ \text{col}_5+(-1)^{5-2}\,\pi\,(5-1,2)\ \text{col}_4$
$+(-1)^{5-3}\,\pi\,(5-1,3)\ \text{col}_3+(-1)^{5-4}\,\pi\,(5-1,4)\ \text{col}_2$
$+(-1)^{5-5}\,\pi\,(5-1,5)\ \text{col}_1$ where $\pi\,(n,r)$ denotes the product $\{n(n-1)\,(n-2)\ldots r\}$, $n-r$ being a positive integer or zero and the product being taken as unity when r is greater than n

then we have $D_5 = \dfrac{9(x-1)}{24} \begin{vmatrix} x-5 & -1 & 0 & 0 & 1 \\ 24, & 3x & -1 & 0 & -12 \\ & 21 & 5x & -1 & 60 \\ & & 16 & 7x & -168 \\ & & & 1 & 24 \end{vmatrix}$

Then by the operation

$2\times \mathrm{col}_5 + (-1)^{5-2}\,\pi\,(5-2,\,1)\,(2\cdot5-2)\,\mathrm{col}_4$

$\quad + (-1)^{5-3}\,\pi\,(5-2,\,2)\,(2\cdot5-3)\,\mathrm{col}_3$

$\quad + (-1)^{5-4}\,\pi\,(5-2,\,3)\,(2\cdot5-4)\,\mathrm{col}_2$

$\quad + (-1)^{5-5}\,\pi\,(5-2,\,4)\,(2\cdot5-5)\,\mathrm{col}_1$, we obtain

$$D_5 = -3\,(x-1)^2 \begin{vmatrix} x-5, & -1, & 0, & 5 \\ 24, & 3x, & -1, & -54 \\ & 21, & 5x & 210 \\ & & 1 & -21 \end{vmatrix}$$

Now performing the operation

$2\times 2\,\mathrm{col}_4 + (-1)^{5-3}\,\pi\,(5-3,\,1)\,\pi\,(2\cdot5-3,\,2\cdot5-4)\,\mathrm{col}_3$

$\quad + (-1)^{5-4}\,\pi\,(5-3,\,2)\,\pi\,(2\cdot5-4,\,2\cdot5-5)\,\mathrm{col}_2$

$\quad + (-1)^{5-5}\,\pi\,(5-3,\,3)\,\pi\,(2\cdot5-5,\,2\cdot5-6)\,\mathrm{col}_1$, we have

$$D_5 = \frac{21\times 3\,(x-1)^3}{4} \begin{vmatrix} x-5, & -1, & 20 \\ 24, & 3x, & -180 \\ & 1 & 20 \end{vmatrix}$$

On this last determinant perform the operation

$2\times 3\,\mathrm{col}_3 + (-1)^{5-4}\,\pi\,(5-4,\,1)\,\pi\,(2\cdot5-4,\,2\cdot5-6)\,\mathrm{col}_2$

$\quad + (-1)^{5-5}\,\pi\,(5-4,\,2)\,\pi\,(2\cdot5-5,\,2\cdot5-7)\,\mathrm{col}_1$ then we obtain

$$D_5 = -7\times 9\,(x-1)^4 \begin{vmatrix} x-5, & 60 \\ 1 & -15 \end{vmatrix}$$

On this again perform the operation

$2\times 4\,\mathrm{col}_2 + (-1)^{5-5}\,\pi\,(5-5,\,1)\,\pi\,(2\cdot5-5,\,2\cdot5-8)\,\mathrm{col}_1$

Then we have $D_3 = \dfrac{-7\times 9\ (x-1)^5}{8} \begin{vmatrix} x-5, & 120 \\ 1 & 0 \end{vmatrix} = 1\times 3\times 5\times 7 \times 9\ (x-1)^5$

Similarly we can show than $D_n = 1\cdot 3\cdot 5\ldots(2n-1)\ (x-1)^n$.

6. In the case of D_n, if no factor is removed from the last column of the determinant that results from any operation, we shall have to perform the following operations :—

$$(-1)^{n-1}\ \pi\ (n-1, 1)\ \mathrm{col}_n + (-1)^{n-2}\ \pi\ (n-1, 2)\ \mathrm{col}_{n-1}$$
$$+(-1)^{n-3}\ \pi\ (n-1, 3)\ \mathrm{col}_{n-2} + \ldots + (n-1)\ (n-2)\ \mathrm{col}_3$$
$$-(n-1)\ \mathrm{col}_2 + \mathrm{col}_1 = \mathrm{col}_n{}^{(1)} \quad \ldots \quad (a_1)$$

$$2\ \mathrm{col}_n{}^{(1)} + (-1)^{n-2}\ \pi\ (n-2, 1)\ (2n-2)\ (x-1)\ \mathrm{col}_{n-1}$$
$$+(-1)^{n-3}\ \pi\ (n-2, 2)\ (2n-3)\ (x-1)\ \mathrm{col}_{n-2} + \ldots$$
$$+(n-2)\ (n-3)\ (n+2)\ (x-1)\ \mathrm{col}_3$$
$$-(n-2)\ (n+1)\ (x-1)\ \mathrm{col}_2 + n\ (x-1)\ \mathrm{col}_1 = \mathrm{col}_n{}^{(2)} \ldots \quad (a_2)$$

$$2\times 2\ \mathrm{col}_n{}^{(2)} + 0\ \mathrm{col}_{n-1} + (-1)^{n-3}\ \pi\ (n-3, 1)\ \pi\ (2n-3, 2n-4)$$
$$\times (x-1)^2\ \mathrm{col}_{n-2} + (-1)^{n-4}\ \pi\ (n-3, 2)\ \pi\ (2n-4,$$
$$2n-5)\ (x-1)^2\ \mathrm{col}_{n-3} + \ldots + n\ (n-1)\ (x-1)^2\ \mathrm{col}_1$$
$$= \mathrm{col}_n{}^{(3)} \quad \ldots \quad (a_3)$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$2\ (n-1)\ \mathrm{col}_n{}^{(n-1)} + 0\ \mathrm{col}_{n-1} + 0\ \mathrm{col}_{n-2} + \ldots + 0\ \mathrm{col}_2$$
$$+\{2\times 3\times 4\ldots(n-1)\ n\}\ (x-1)^n\ \mathrm{col}_1 = \mathrm{col}_n{}^{(n)} \quad \ldots \quad (a_n)$$

Since any two operatiens of the type

$$A_1\ \mathrm{col}_3 + A_2\ \mathrm{col}_2 + A_3\ \mathrm{col}_1 = \mathrm{col}_3{}^{(1)}$$
$$B_1\ \mathrm{col}_3{}^{(1)} + B_2\ \mathrm{col}_2 + B_3\ \mathrm{col}_1 = \mathrm{col}_3{}^{(2)}$$

may be substituted by the single operation

$$A_1\ B_1\ \mathrm{col}_3 + (B_1\ A_2 + B_2)\ \mathrm{col}_2 + (B_1\ A_3 + B_3)\ \mathrm{col}_1 = \mathrm{col}_3{}^{(1)}$$

we may substitute for the operations $(a_1, a_2 \ldots a_n)$ a single operation in which the multiplier of $\mathrm{col}_{r-(r-1)}$ will be

$$(-1)^{n-r}\ 2^{n-1}\ \underline{|n-1}\ [2^{r-1}\ \pi\ (n-1, r)$$
$$+2^{r-2}\ \pi\ (n-2, r-1)\ (2n-r)\ (x-1)$$

$$+2^{r-3}\frac{1}{|\underline{2}}\,\pi\,(n-3,\,r-2)\,\pi\,(2n-r,\,2n-r-1)\,(x-1)^2$$

$$+2^{r-4}\frac{1}{|\underline{3}}\,\pi\,(n-4,\,r-3)\,\pi\,(2n-r,\,2n-r-2)\,(x-1)^3+\ldots$$

$$+2\cdot\frac{1}{|\underline{r-2}}\,\pi\,(n-r+1,\,2)\,\pi\,(2n-r,\,2n-2r+3)\,(x-1)^{r-2}$$

$$+\frac{1}{|\underline{r-1}}\,\pi\,(n-r,\,1)\,\pi\,(2n-r,\,2n-2r+2)\,(x-1)^{r-1}\Big]$$

Rejecting the factor $|\underline{n-1}$ which is common to all the multipliers and writing in the reverse order we have the multiplier of $\mathrm{col}_{n-(r-1)}$

$$=(-1)^{n-r}\,2^{n-r}\Big[\frac{1}{|\underline{r-1}}\,\pi\,(n-r,\,1)\,\pi\,(2n-r,\,2n+2r+2)\,(x-1)^{r-1}$$

$$+\frac{2}{|\underline{r-2}}\,\pi\,(n-r+1,\,2)\,\pi\,(2n-r,\,2n-2r+3)\,(x-1)^{r-2}$$

$$+\frac{2^2}{|\underline{r-3}}\,\pi\,(n-r+2,\,3)\,\pi\,(2n-r,\,2n-2r+4)\,(x-1)^{r-3}$$

$$+\ldots+2^{r-2}\frac{1}{|\underline{1}}\,\pi\,(n-2,\,r-1)\,(2n-r)\,(x-1)+2^{r-1}\,\pi\,(n-1,\,r)\Big]$$

$$=(-1)^{n-r}\,2^{n-r}\,k\Big[\frac{1}{|\underline{r-1}}\cdot(x-1)^{r-1}+\frac{1}{|\underline{r-2}\,|\underline{1}}\cdot(x-1)^{r-2}$$

$$+\frac{1}{|\underline{r-3}\,|\underline{2}}\,\frac{a+1}{a}\,(x-1)^{r-3}+\frac{1}{|\underline{r-4}\,|\underline{3}}\,\frac{a+3}{a}\,(x-1)^{r-4}$$

$$+\frac{1}{|\underline{r-5}\,|\underline{4}}\,\frac{(a+3)\,(a+5)}{a\,(a+2)}\,(x-1)^{r-5}+\ldots$$

$$+\frac{1}{|\underline{r-2p}\,|\underline{2p-1}}\,\frac{\{(a+2p-1)\,(a+2p+1)\ldots(a+4p-5)\}}{\{a\,(a+2)\,(a+4)\ldots(a+2p-4)\}}\,(x-1)^{r-2p}$$

$$+\frac{1}{|\underline{r-2p-1}\,|\underline{2p}}\,\frac{\{(a+2p-1)\,(a+2p+1)\ldots(a+4p-3)\}}{\{a\,(a+2)\,(a+4)\ldots(a+2p-2)\}}$$

$$\times(x-1)^{r-(2p+1)}+\ldots\Big]$$

where $k=\pi(n-r, 1)\,\pi(2n-r, 2n-2r+2)$ and $a=2n-2r+3$.

Hence the multiplier of $\mathrm{col}_{n-(r-1)}$ is

$$(-1)^{n-r}\,2^{n-r}\,k\left[\frac{1}{\underline{|r-1}}\,x^{r-1}-\frac{1}{\underline{|r-2}}\left\{\frac{1}{\underline{|1}}-\frac{1}{\underline{|1}}\right\}x^{r-2}\right.$$

$$+\frac{1}{\underline{|r-3}}\left\{\frac{1}{\underline{|2}}-\frac{1}{\underline{|1}\,\underline{|1}}+\frac{1}{\underline{|2}}\,\frac{a+1}{a}\right\}x^{r-3}$$

$$-\frac{1}{\underline{|r-4}}\left\{\frac{1}{\underline{|3}}-\frac{1}{\underline{|2}\,\underline{|1}}+\frac{1}{\underline{|1}\,\underline{|2}}\,\frac{a+1}{a}-\frac{1}{\underline{|3}}\,\frac{a+3}{a}\right\}x^{r-4}+\dots$$

$$-\frac{1}{\underline{|r-2p}}\left\{\frac{1}{\underline{|2p-1}}-\frac{1}{\underline{|2p-2}\,\underline{|1}}+\frac{1}{\underline{|2p-3}\,\underline{|2}}\,\frac{a+1}{a}-\dots\right.$$

$$\left.-\frac{1}{\underline{|2p-1}}\,\frac{(a+2p-1)\,(a+2p+1)\dots(a+4p-5)}{a\,(a+2)\dots(a+2p-4)}\right\}x^{r-2p}$$

$$+\frac{1}{\underline{|r-2p-1}}\left\{\frac{1}{\underline{|2p}}-\frac{1}{\underline{|2p-1}\,\underline{|1}}+\frac{1}{\underline{|2p-2}\,\underline{|2}}\,\frac{a+1}{a}\right.$$

$$\left.-\frac{1}{\underline{|2p-3}\,\underline{|3}}\,\frac{a+3}{a}+\dots+\frac{1}{\underline{|2p}}\,\frac{(a+2p-1)\,(a+2p+1)\dots(a+4p-3)}{a\,(a+2)\dots(a+2p-2)}\right\}$$

$$\left.\times\,x^{r-(2p+1)}\;-\;\dots\right]$$

So by theorem (6), the multiplier of the $\mathrm{col}_{n-(r-1)}$

$$=(-1)^{n-r}\,2^{n-r}\,k\left[\frac{1}{\underline{|r-1}}\,x^{r-1}+\frac{1}{\underline{|r-3}\,\underline{|2}}\,\frac{1}{a}\,x^{r-3}\right.$$

$$+\frac{1\cdot3}{\underline{|r-5}\,\underline{|4}}\,\frac{1}{a\,(a+2)}\,x^{r-5}+\frac{1\cdot3\cdot5}{\underline{|r-7}\,\underline{|6}}\,\frac{1}{a\,(a+2)\,(a+4)}\,x^{r-7}$$

$$\left.+\dots+\frac{1\cdot3\cdot5\dots(2p-1)}{\underline{|r-(2p+1)}\,\underline{|2p}\,a\,(a+2)\dots(a+2p-2)}\,x^{r-(2p+1)}\;+\dots\right]\dots\;(9)$$

(i) To illustrate the application of the formula given in (9), let us consider the continuant of the 7th order (i.e $n=7$) then the multiplier of $\mathrm{col}_7=(-1)^6\,2^6\,\pi(6,1)\,\pi(13,14)=46080$ for $\pi(13,14)=1$ (where $r=1$)

$$\dots\qquad\dots\quad \mathrm{col}_6=(-1)^5\;2^5\;\pi\;(5,\,1)\;\pi\;(12,\,12)\left[\frac{1}{\underline{|1}}\,x\right]$$

$=-46080\,x$ for $\pi\;(12,\,12)=12$, (where $r=2$)

... ...

$$\text{multiplier of } \mathrm{col}_1 = \pi(7,2)\left[\frac{1}{\underline{|6}}x^6 + \frac{1}{\underline{|4}\,\underline{|2}\cdot 3}x^4 + \frac{1\cdot 3}{\underline{|2}\,\underline{|4}\cdot 3\cdot 5}x^2 + \right.$$

$$\left. + \frac{1\cdot 3\cdot 5}{\underline{|6}\cdot 3\cdot 5\cdot 7}\right], \text{ (where } r=7)$$

$$=7x^6+35x^4+21x^2+1$$

Hence the operation in this case is

$$46080\,\mathrm{col}_7 - 46080x\,\mathrm{col}_6 + (21120x^2+1920)\,\mathrm{col}_5 - (5760x^3$$
$$+1920x)\,\mathrm{col}_4 + (1008x^4+864x^2+48)\,\mathrm{col}_3 - (112x^5+224x^3$$
$$+48x)\,\mathrm{col}_2 + (7x^6+35x^4+21x^2+1)\,\mathrm{col}_1$$

and the continuant D_7, by this operation, becomes

$$\frac{1}{46080}\begin{vmatrix} x-7, & -1 & 0 & & & & 7(x-1)^7 \\ 48 & 3x & -1 & & & & 0 \\ & 45 & 5x & -1 & & & 0 \\ & & 40 & 7x & -1 & & 0 \\ & & & 33 & 9x & -1 & 0 \\ & & & & 24 & 11x & 0 \\ & & & & & 13 & 0 \end{vmatrix}$$

$$=(-1)^{7-1}\frac{7(x-1)^7\times 48\times 45\times 40\times 33\times 24\times 13}{46080} = 1\times 3\times 5\times 7\times 9 \times 11\times 13\,(x-1)^7$$

7. The continuant

$$\begin{vmatrix} 1, & x & & & & & \\ a+n-1, & a, & x & & & & \\ & 1-n, & a+1, & x & & & \\ & & a(a+n), & a+2, & x & & \\ & \cdots & \cdots & \cdots & \cdots & & \\ & & & & 1-n, & a+2n-3, & x \\ & & & & & a+n-2, & 1 \end{vmatrix}_{2n}$$

$=\{a(a+1)(a+2)\ldots(a+2n-3)\}(1-x)^n$ *i.e.* equal to the product of the principal diagonal terms multiplied by $(1-x)^n$. Here the

elements, except the first and last, of the lower minor diagonal are given by

$$e_{2m}=(a+m-2)\ (a+m+n-2)$$

and

$$e_{2m+1}=m\ (m-n)$$

where e_r denotes the element of this diagonal in the r^{th} row.

Let us first consider the particular case when $n=3$

viz.

$$\begin{vmatrix} 1 & x & & & & \\ a+2, & a, & x & & & \\ & -2, & a+1, & x & & \\ & & a\ (a+3), & a+2, & x & \\ & & & -2 & a+3 & x \\ & & & & a+1 & 1 \end{vmatrix}$$

and on this perform the first operation

$$-2a\ (a+1)\ \text{col}_6+2a\ \text{col}_5+2a\ \text{col}_4-2\ \text{col}_3-\text{col}_2+\text{col}_1$$

This enables us to remove the factor $\dfrac{(1-x)\ (a+1)\ (-2)}{2a\ (a+1)}$ and then subtracting the first column from the last, we can remove another factor a and write the co-factor in the form

$$\begin{vmatrix} 1 & x & 0 & 0 & 0 \\ a+2, & a & x & 0 & -1 \\ & -2 & a+1 & x & -2 \\ & & a\ (a+3) & a+2, & -2 \\ & & & 1 & -(a+1) \end{vmatrix}$$

On this co-factor, performing the 2nd operation

$$\text{col}_5+(a+1)\ \text{col}_4-\text{col}_3-\text{col}_2+\text{col}_1$$

we can remove the factor $(1-x)\ a\ (a+3)$ and get the co-factor

$$\begin{vmatrix} 1 & x & 1 \\ a+2, & a, & 1 \\ & -2 & -(a+1) \end{vmatrix}$$

Then substracting the first column from the last we can remove another factor $(a+1)$ and obtain

$$\begin{vmatrix} 1 & x & 0 \\ a+2, & a, & -1 \\ & -2 & -1 \end{vmatrix}$$

Then performing the 3rd operation $2\ \mathrm{col}_3 - \mathrm{col}_2 + \mathrm{col}_1$, we have

$$\tfrac{1}{2}\begin{vmatrix} 1 & x & 1-x \\ a+2, & a & 0 \\ & -2 & 0 \end{vmatrix} \quad \text{which is equal to } -(a+2)\,(1-x)$$

Thus the result is $a\,(a+1)\,(a+2)\,(a+3)\,(1-x)^3$.

In the general case, if m_k denotes the multiplier of the k^{th} column and l that of the last column, then

in the first operation ... $\begin{cases} m_{2r} = (-1)^{\frac{2r\,(2r-1)}{2}}\ \pi\,(a+r-2,\ a)\ \pi(n-1,\ n-r+1) \\ m_{2r+1} = (-1)^{\frac{(2r+1)\,2r}{2}}\ \pi(a+r-2,\ a)\ \pi(n-1,\ n-r) \\ l \text{ is governed by these two rules.} \end{cases}$

in the second operation $\begin{cases} m_{2r} = (-1)^{\frac{2r\,(2r-1)}{2}}\ \pi(a+r-1,\ a+1)\ \pi(n-2,\ n-r) \\ m_{2r+1} = (-1)^{\frac{(2r+1)\,2r}{2}}\ \pi(a+r-1,\ a+1)\ \pi(n-2,\ n-r-1) \\ l=1 \end{cases}$

in the third operation... $\begin{cases} m_{2r} = (-1)^{\frac{2r\,(2r-1)}{2}}\ \pi(a+r,\ a+2)\ \pi(n-3,\ n-r-1) \\ m_{2r+1} = (-1)^{\frac{(2r+1)\,2r}{2}}\ \pi(a+r,\ a+2)\ \pi(n-3,\ n-r-2) \\ l=2 \end{cases}$

in the fourth operation
$$\begin{cases} m_{2r} = (-1)^{\frac{2r(2r-1)}{2}} \pi(a+r+1,\, a+3) \\ \qquad\qquad\qquad\qquad \pi(n-4,\, n-r-2) \\ m_{2r+1} = (-1)^{\frac{(2r+1)\,2r}{2}} \pi(a+r+1,\, a+3) \\ \qquad\qquad\qquad\qquad \pi(n-4,\, n-r-3) \\ l = 3 \end{cases}$$

and so on.

If is to be noted in this connection that each operation will enable us to remove certain factors and before performing the next operation, these factors are to be removed, the first column to be subtracted from the last and another factor which we can then remove from the last column is to be removed.

8

THE OSCULATING CONIC IN HOMOGENEOUS CO-ORDINATES

BY

GURUDAS BHAR, M.Sc.

The object of the present paper is to deduce in elegant forms the equation of the osculating conic at any point of a curve whose equation is given as a homogeneous equation of a given degree in ξ, η, ζ where ξ, η, ζ are a given system of trilinear co-ordinates satisfying a given identical relation

$$l\xi+m\eta+n\zeta=1.$$

The co-ordinates α, β, γ of any point of the given curve will be supposed expressible as functions of any arbitrary parameter t by

$$\alpha=f_1(t),\ \beta=f_2(t),\ \gamma=f_3(t);$$

or more generally by

$$\alpha:\beta:\gamma=f_1(t):f_2(t):f_3(t).$$

In this case the absolute co-ordinates α, β, γ may be written as

$$\alpha=\frac{f_1(t)}{\theta},\ \beta=\frac{f_2(t)}{\theta},\ \gamma=\frac{f_3(t)}{\theta},$$

where

$$\theta=lf_1(t)+mf_2(t)+nf_3(t).$$

We shall say in this latter case that α, β, γ are *relative homogeneous co-ordinates* whereas in the former case we shall call them *absolute homogeneous co-ordinates* of a point on the curve.

1

If we take *absolute* homogeneous co-ordinates in a given system of trilinears then at a point α, β, γ of the given curve which corresponds to a certain value of t, the equation to the osculating conic can evidently

be written in the form

$$\begin{vmatrix} \xi^2 & \eta^2 & \zeta^2 & \eta\zeta & \zeta\xi & \xi\eta \\ \alpha^2 & \beta^2 & \gamma^2 & \beta\gamma & \gamma\alpha & \alpha\beta \\ D(\alpha^2) & D(\beta^2) & D(\gamma^2) & D(\beta\gamma) & D(\gamma\alpha) & D(\alpha\beta) \\ D^2(\alpha^2) & D^2(\beta^2) & D^2(\gamma^2) & D^2(\beta\gamma) & D^2(\gamma\alpha) & D^2(\alpha\beta) \\ D^3(\alpha^2) & D^3(\beta^2) & D^3(\gamma^2) & D^3(\beta\gamma) & D^3(\gamma\alpha) & D^3(\alpha\beta) \\ D^4(\alpha^2) & D^4(\beta^2) & D^4(\gamma^2) & D^4(\beta\gamma) & D^4(\gamma\alpha) & D^4(\alpha\beta) \end{vmatrix} = 0 \quad \ldots \quad (1)$$

where D^n denotes $\left(\dfrac{d}{dt}\right)^n$.

It is *important* to notice that we shall get the equation to the osculating conic in the *same form* even if we take α, β, γ to be *relative* co-ordinates of a point on the curve. For the second system is derivable from the first by writing $\theta\alpha$, $\theta\beta$, $\theta\gamma$ for α, β, γ respectively and all the θ's and their differential co-efficients will be eliminated from the equation on simplification.

2

Let us now take

$$U \equiv \begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix},$$

$$V \equiv \begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}$$

$$W \equiv \begin{vmatrix} \xi & \eta & \xi \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix};$$

also let

$$\Delta_{12} \equiv \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix},$$

$$\Delta_{13} \equiv \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha''' & \beta''' & \gamma''' \end{vmatrix}$$

and generally

$$\Delta_{mn} \equiv \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^{(m)} & \beta^{(m)} & \gamma^{(m)} \\ \alpha^{(n)} & \beta^{(n)} & \gamma^{(n)} \end{vmatrix}$$

where $\alpha^{(m)} = D^m(\alpha) = \left(\dfrac{d}{dt}\right)^m (\alpha)$.

The equation (1) to the osculating conic in the new homogeneous co-ordinates U, V, W may evidently be written as

$$\begin{vmatrix} U^2 & UW & V^2 & UV & VW & W^2 \\ 0 & 0 & 0 & 0 & 0 & \Delta_{12}^2 \\ 0 & 0 & 0 & 0 & -\Delta_{12}^2 & 0 \\ 0 & \Delta_{12}^2 & 2\Delta_{12}^2 & 0 & 0 & 0 \\ 0 & \Delta_{12}\,\Delta_{13} & 0 & -3\Delta_{12}^2 & \Delta_{12}\,\Delta_{23} & 0 \\ 6\,\Delta_{12}^2 & \Delta_{12}\,\Delta_{14} & -8\Delta_{12}\,\Delta_{23} & -4\Delta_{12}\,\Delta_{13} & \Delta_{12}\,\Delta_{24} & 0 \end{vmatrix} = 0 \qquad (2)$$

which reduces to

$$\begin{vmatrix} U^2 & UW & V^2 & UV \\ 0 & \Delta_{12}^2 & 2\Delta_{12}^2 & 0 \\ 0 & \Delta_{12}\,\Delta_{13} & 0 & -3\Delta_{12}^2 \\ 6\Delta_{12}^2 & \Delta_{12}\,\Delta_{14} & -8\Delta_{12}\,\Delta_{23} & -4\Delta_{12}\,\Delta_{13} \end{vmatrix} = 0,$$

$$\text{or } U^2 \begin{vmatrix} 1 & 0 & 0 \\ \Delta_{13} & \Delta_{13} & 3\Delta_{12} \\ \Delta_{14} & 4\Delta_{23}+\Delta_{14} & 4\Delta_{13} \end{vmatrix} = 3\Delta_{12} \begin{vmatrix} UW & V^2 & UV \\ 1 & 2 & 0 \\ \Delta_{13} & 0 & -3\Delta_{12} \end{vmatrix}$$

or $\quad (12\,\Delta_{12}\,\Delta_{23} - 4\Delta_{13}^2 + 3\Delta_{12}\,\Delta_{14})\,U^2$

$$= 3\Delta_{12}\,(6\Delta_{12}\,UW - 3\Delta_{12}\,V^2 + 2\Delta_{13}\,UV),$$

or $(\Delta_{13} \mathrm{U}-3\Delta_{12} \mathrm{V})^2+(12\Delta_{12} \Delta_{23}-5\Delta_{13}^2+3\Delta_{12} \Delta_{14}) \mathrm{U}^2$

$$=18\Delta_{12}^2 \mathrm{UW},$$

or $(\Delta_{13} \mathrm{U}-3\Delta_{12} \mathrm{V})^2+\sigma \mathrm{U}^2=18\Delta_{12}^2 \mathrm{UW}$ ... (3)

where

$$\sigma=12\Delta_{12} \Delta_{23}-5\Delta_{13}^2+3\Delta_{12} \Delta_{14}.$$

It can also be written as

$$(\Delta_{13} \mathrm{U}-3\Delta_{12} \mathrm{V})^2+\sigma \left(\mathrm{U}-9 \frac{\Delta_{12}^2}{\sigma} \mathrm{W}\right)^2-81\frac{\Delta_{12}^4}{\sigma} \mathrm{W}^2=0 \quad \ldots \quad (4)$$

From (4) it is evident that

$$\left.\begin{aligned} &\Delta_{13} \mathrm{U}-3\Delta_{12} \mathrm{V}=0, \\ &\mathrm{U}-9 \frac{\Delta_{12}^2}{\sigma} \mathrm{W}=0, \\ &\mathrm{W}=0 \end{aligned}\right\} \quad (5)$$

form a self-conjugate triangle of the osculating conic.

Now the tangent to the curve and therefore also to the osculating conic at α, β, γ is evidently

$$\mathrm{U} \equiv \begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix} =0 \quad \ldots \quad (6)$$

whether α, β, γ be absolute or relative homogeneous co-ordinates.

Again if we take α, β, γ to be *absolute* homogeneous co-ordinates, then

$$l\alpha+m\beta+n\gamma=1;$$

therefore

$$\left.\begin{aligned} l\alpha'+m\beta'+n\gamma'=0, \\ l\alpha''+m\beta''+n\gamma''=0 \end{aligned}\right\} \quad \ldots \quad (7)$$

The equation to the line at infinity is

$$l\xi+m\eta+n\zeta=0 \quad \ldots \quad (8)$$

Eliminating l, m, n from (7) and (8) we have for the equation of the line at infinity

$$W \equiv \begin{vmatrix} \xi & \eta & \zeta \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} = 0 \qquad \dots \quad (9)$$

The pole of the line at infinity is the *centre* of the osculating conic which is determined by the intersection of

$$\Delta_{13}\, U - 3\Delta_{12}\, V = 0$$

and

$$U - 9\,\frac{\Delta_{12}^2}{\sigma}\, W = 0.$$

But $U=0$ is the tangent to the osculating conic at the point of contact α, β, γ. Therefore

$$U - 9\,\frac{\Delta_{12}^2}{\sigma}\, W = 0 \qquad \dots \quad (10)$$

is the diameter of the osculating conic parallel to the tangent.

Again $U=0$ and $V=0$ both pass through α, β, γ. Therefore

$$\Delta_{13}\, U - 3\Delta_{12}\, V = 0 \qquad \dots \quad (11)$$

is the diameter of the osculating conic passing through the point of contact. It is the *axis of aberrancy* of the given curve at α, β, γ.

3

The co-ordinates of the centre are given by the intersection of

$$\Delta_{13}\, U - 3\Delta_{12}\, V = 0,$$

and

$$U - 9\,\frac{\Delta_{12}^2}{\sigma}\, W = 0;$$

that is by

$$\frac{U}{9\Delta_{12}^2} = \frac{V}{3\Delta_{12}\,\Delta_{13}} = \frac{W}{\sigma},$$

or by

$$\frac{\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}}{9\Delta_{12}^2}=\frac{\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}}{3\Delta_{12}\,\Delta_{13}}=\frac{\begin{vmatrix} \xi & \eta & \zeta \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}}{\sigma} \quad \ldots \quad (12)$$

Now from (7) we have

$$\frac{l}{\beta'\gamma''-\beta''\gamma'}=\frac{m}{\gamma'\alpha''-\gamma''\alpha'}=\frac{n}{\alpha'\beta''-\alpha''\beta'}$$

$$=\frac{l\alpha+m\beta+n\gamma}{\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}}=\frac{1}{\Delta_{12}}.$$

Hence

$$\left.\begin{aligned} \beta'\gamma''-\beta''\gamma' &= l\Delta_{12}, \\ \gamma'\alpha''-\gamma''\alpha' &= m\Delta_{12}; \\ \alpha'\beta''-\alpha''\beta' &= n\Delta_{12}. \end{aligned}\right\} \quad \ldots \quad (13)$$

Therefore the determinant

$$\begin{vmatrix} \xi & \eta & \zeta \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}=(\beta'\gamma''-\beta''\gamma')\,\xi+(\gamma'\alpha''-\gamma''\alpha')\,\eta+(\alpha'\beta''-\alpha''\beta')\,\zeta,$$

$$=\Delta_{12}\,(l\xi+m\eta+n\zeta),\ \text{by (13)},$$

$$=\Delta_{12} \quad \ldots \quad (14)$$

Each of the ratios in (12) is therefore equal to $\frac{\Delta_{12}}{\sigma}$ and the co-ordinates of the centre are determined by

$$(\beta\gamma'-\beta'\gamma)\,\xi+(\gamma\alpha'-\gamma'\alpha)\,\eta+(\alpha\beta'-\alpha'\beta)\,\zeta=9\,\frac{\Delta_{12}^3}{\sigma}.$$

$$(\beta\gamma''-\beta''\gamma)\,\xi+(\gamma\alpha''-\gamma''\alpha)\,\eta+(\alpha\beta''-\alpha''\beta)\,\zeta=3\,\frac{\Delta_{12}^2\,\Delta_{13}}{\sigma}.$$

$$(\beta'\gamma''-\beta''\gamma')\,\xi+(\gamma'\alpha''-\gamma''\alpha')\eta+(\alpha'\beta''-\alpha''\beta')\,\zeta=\Delta_{12},$$

whence we get for the co-ordinates of the centre

$$\left.\begin{aligned}
\xi&=\alpha-\frac{3\Delta_{12}(\Delta_{13}\,\alpha'-3\Delta_{12}\,\alpha'')}{\sigma},\\
\eta&=\beta-\frac{3\Delta_{12}(\Delta_{13}\,\beta'-3\Delta_{12}\,\beta'')}{\sigma},\\
\zeta&=\gamma-\frac{3\Delta_{12}(\Delta_{13}\,\gamma'-3\Delta_{12}\,\gamma'')}{\sigma}.
\end{aligned}\right\}\qquad\ldots\quad(15)$$

4

If the osculating conic passes through six consecutive points at α, β, γ, we must have

$$\begin{vmatrix}
\alpha^2 & \beta^2 & \gamma^2 & \beta\gamma & \gamma\alpha & \alpha\beta\\
D(\alpha^2) & D(\beta^2) & D(\gamma^2) & D(\beta\gamma) & D(\gamma\alpha) & D(\alpha\beta)\\
D^2(\alpha^2) & D^2(\beta^2) & D^2(\gamma^2) & D^2(\beta\gamma) & D^2(\gamma\alpha) & D^2(\alpha\beta)\\
D^3(\alpha^2) & D^3(\beta^2) & D^3(\gamma^2) & D^3(\beta\gamma) & D^3(\gamma\alpha) & D^3(\alpha\beta)\\
D^4(\alpha^2) & D^4(\beta^2) & D^4(\gamma^2) & D^4(\beta\gamma) & D^4(\gamma\alpha) & D^4(\alpha\beta)\\
D^5(\alpha^2) & D^5(\beta^2) & D^5(\gamma^2) & D^5(\beta\gamma) & D^5(\gamma\alpha) & D^5(\alpha\beta)
\end{vmatrix}=0.\qquad\ldots\quad(16)$$

From this we derive, just as (2) is derived from (1), the relation

$$\begin{vmatrix}
0 & 0 & 0 & 0 & 0 & \Delta_{12}^2\\
0 & 0 & 0 & 0 & -\Delta_{12}^2 & 0\\
0 & \Delta_{12}^2 & 2\Delta_{12}^2 & 0 & 0 & 0\\
0 & \Delta_{12}\Delta_{13} & 0 & -3\Delta_{12}^2 & \Delta_{12}\Delta_{23} & 0\\
6\Delta_{12}^2 & \Delta_{12}\Delta_{14} & -8\Delta_{12}\Delta_{23} & -4\Delta_{12}\Delta_{13} & \Delta_{12}\Delta_{24} & 0\\
20\Delta_{12}\Delta_{13} & \Delta_{12}\Delta_{15}+10\Delta_{12}\Delta_{123} & -10\Delta_{12}\Delta_{24} & -5\Delta_{12}\Delta_{14}+10\Delta_{12}\Delta_{23} & \Delta_{12}\Delta_{25} & 0
\end{vmatrix}=0$$

where

$$\Delta_{123}=\begin{vmatrix}
\alpha' & \beta' & \gamma'\\
\alpha'' & \beta'' & \gamma''\\
\alpha''' & \beta''' & \gamma'''
\end{vmatrix};$$

it reduces to

$$20\Delta_{12}\Delta_{13}\begin{vmatrix}
1 & 0 & 0\\
\Delta_{13} & \Delta_{13} & 3\Delta_{12}\\
\Delta_{14} & \Delta_{14}+4\Delta_{23} & 4\Delta_{13}
\end{vmatrix}.$$

$$=3\Delta_{12}\begin{vmatrix} \Delta_{12}\Delta_{15}+10\Delta_{12}\Delta_{123} & -10\Delta_{12}\Delta_{24} & -5\Delta_{12}\Delta_{14}+10\Delta_{12}\Delta_{23} \\ 1 & 2 & 0 \\ \Delta_{13} & 0 & -3\Delta_{12} \end{vmatrix}$$

On further simplification we obtain

$$40\Delta_{13}^3-45\Delta_{12}\Delta_{13}\Delta_{14}+9\Delta_{12}^2\Delta_{15}-90\Delta_{12}\Delta_{13}\Delta_{23}$$
$$+45\Delta_{12}^2\Delta_{24}+90\Delta_{12}^2\Delta_{123}=0 \quad \ldots \quad (17)$$

which is therefore the condition that α, β, γ may be a sextactic point on the curve.

If α, β, γ be *absolute* homogeneous co-ordinates $\Delta_{123}=0$ and the above condition reduces to

$$40\Delta_{13}^3-45\Delta_{12}\Delta_{13}\Delta_{14}+9\Delta_{12}^2\Delta_{15}-90\Delta_{12}\Delta_{13}\Delta_{23}$$
$$+45\Delta_{12}^2\Delta_{24}=0 \quad \ldots \quad (18)$$

5

Suppose now α, β, γ are the relative homogeneous co-ordinates of a point on the curve. Let us write

$$\Theta=\mathrm{L}\xi+\mathrm{M}\eta+\mathrm{N}\zeta$$

where L, M, N are any constants ; also let

$$\theta=\mathrm{L}\alpha+\mathrm{M}\beta+\mathrm{N}\gamma,$$

so that

$$\theta'=\mathrm{L}\alpha'+\mathrm{M}\beta'+\mathrm{N}\gamma',$$

$$\theta''=\mathrm{L}\alpha''+\mathrm{M}\beta''+\mathrm{N}\gamma''$$

and generally

$$\theta^{(n)}=\mathrm{L}\alpha^{(n)}+\mathrm{M}\beta^{(n)}+\mathrm{N}\gamma^{(n)}.$$

The equation (1) to the osculating conic can now be written as

$$\begin{vmatrix} \mathrm{U}\Theta & \mathrm{V}^2 & \mathrm{UV} & \mathrm{V}\Theta & \Theta^2 \\ 0 & 0 & 0 & 0 & \theta^2 \\ 0 & 0 & 0 & -\theta\Delta_{12} & 2\theta\theta' \\ \theta\Delta_{12} & 2\Delta_{12}^2 & 0 & -2\theta\Delta_{12} & 2\theta'^2+2\theta\theta'' \\ 3\theta'\Delta_{12}+\theta\Delta_{13} & 0 & -3\Delta_{12}^2 & \theta\Delta_{22}-3\theta''\Delta_{12} & 6\theta'\theta''+2\theta\theta''' \\ 6\theta''\Delta_{12}+4\theta'\Delta_{13}+\theta\Delta_{14} & -8\Delta_{12}\Delta_{23} & -4\Delta_{12}\Delta_{13} & \theta\Delta_{24}+4\theta'\Delta_{23}-4\theta'''\Delta_{12} & 6\theta''^2+8\theta'\theta'''+2\theta\theta^{iv} \end{vmatrix}=0 \quad (19)$$

On simplification this reduces to

$$(12\theta\Delta_{12}\Delta_{23}+3\theta\Delta_{12}\Delta_{14}-4\theta\Delta_{13}^2+18\theta''\Delta_{12}^2)\,U^2$$

$$+9\theta\Delta_{12}^2\,V^2-18\Delta_{12}^3\,U\Theta-(18\theta'\Delta_{12}^2+6\theta\Delta_{12}\Delta_{13})UV=0,$$

or $$\{(\theta\Delta_{13}+3\theta'\Delta_{12})U-3\theta\Delta_{12}\,V\}^2$$

$$+\{(12\Delta_{12}\Delta_{23}-5\Delta_{13}^2+3\Delta_{12}\Delta_{14})\theta^2-6\theta\theta'\Delta_{12}\Delta_{13}+9\Delta_{12}^2\,(2\theta\theta''-\theta'^2)\}\,U^2$$

$$=18\theta\Delta_{12}^3\,U\Theta,$$

or $$\{(\theta\Delta_{13}+3\theta'\Delta_{12})\,U-3\theta\Delta_{12}\,V\}^2+U^2\,\Sigma=18\theta\Delta_{12}^3\,U\Theta, \qquad \ldots \quad (20)$$

where

$$\Sigma=(12\Delta_{12}\Delta_{23}-5\Delta_{13}^2+3\Delta_{12}\Delta_{14})\theta^2-6\theta\theta'\Delta_{12}\Delta_{13}+9\Delta_{12}^2\,(2\theta\theta''-\theta'^2),$$

$$=\sigma\theta^2-6\theta\theta'\Delta_{12}\Delta_{13}+9\Delta_{12}^2\,(2\theta\theta''-\theta'^2) \qquad \ldots \quad (21)$$

If $\Sigma=0$, the equation to the osculating conic reduces to

$$\{(\theta\Delta_{13}+3\theta'\Delta_{12})\,U-3\theta\Delta_{12}\,V\}^2=18\theta\Delta_{12}^3\,U\Theta \qquad \ldots \quad (22)$$

The form of the equation shows that $U=0$ as well as $\Theta=0$ are tangential to the osculating conic and

$$(\theta\Delta_{13}+3\theta'\Delta_{12})\,U=3\theta\Delta_{12}\,V$$

is the chord of contact. Hence the tangential equation of the osculating conic to the given curve at α, β, γ is

$$\Sigma=0 \qquad \ldots \quad (23)$$

It may be observed that $\Sigma=0$ is a homogeneous equation of the second degree in L, M, N for θ, θ' and θ'' are linear expressions of the first degree in L, M, N.

6

We now proceed to obtain the same results by a direct transformation of the results given by Prof. S. Mukhopadhyaya in his "General Theory of Osculating Conics"—Second paper. [*Journal and Proceedings, Asiatic Society of Bengal (New series), Vol. IV, No. 10, 1908*].

Suppose ξ, η, ζ are absolute homogeneous co-ordinates in the system of trilinaers

$$\left.\begin{aligned}\xi&=AX+BY+C=\lambda\,(p-X\cos\theta-Y\sin\theta)=0,\\ \eta&=A'X+B'Y+C'=\mu\,(q-X\cos\phi-Y\sin\phi)=0,\\ \zeta&=A''X+B''Y+C''=\nu\,(r-X\cos\psi-Y\sin\psi)=0\end{aligned}\right\}\quad\ldots\qquad(24)$$

The identical relation satisfied by ξ, η, ζ is

$$l\xi+m\eta+n\zeta=1 \qquad \ldots \quad (25)$$

where

$$l=\frac{a}{2\lambda\Delta},\ m=\frac{b}{2\mu\Delta},\ n=\frac{c}{2\nu\Delta} \qquad \ldots \quad (26)$$

a, b, c being the sides and Δ the area of the the fundamental triangle.

We shall denote by ξ, η, ζ the current homogeneous co-ordinates in this system and by α, β, γ the co-ordinates of a given point. Similarly X, Y shall denote the current co-ordinates in a rectangular cartesian system and x, y the co-ordinates of a given point in this system.

Let

$$D\equiv\begin{vmatrix}\xi & \eta & \zeta\\ A & A' & A''\\ B & B' & B''\end{vmatrix};$$

then D has a constant value; for

$$D=\xi\,(A'B''-A''B')+\eta\,(A''B-AB'')+\zeta\,(AB'-A'B).$$

$$=\mu\nu\xi\,(\cos\phi\sin\psi-\cos\psi\sin\phi)+\nu\lambda\,\eta\,(\cos\psi\sin\theta-\cos\theta\sin\psi)$$

$$+\lambda\mu\zeta\,(\cos\theta\sin\phi-\cos\phi\sin\theta),$$

$$=\mu\nu\xi\sin\widehat{bc}+\nu\lambda\eta\sin\widehat{ca}+\lambda\mu\zeta\sin\widehat{ab},$$

$\widehat{bc}$, $\widehat{ca}$, $\widehat{ab}$ being the angles between b and c, c and a, a and b respectively. If now ρ be the radius of the circum-circle of the fundamental triangle, we have

$$D=\frac{\lambda\mu\nu}{2\rho}\left\{\frac{a\xi}{\lambda}+\frac{b\eta}{\mu}+\frac{c\zeta}{\nu}\right\}$$

$$=\lambda\mu\nu\,\frac{\Delta}{\rho},\text{ by (25) and (26).}$$

It is therefore evident that

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \mathrm{A} & \mathrm{A}' & \mathrm{A}'' \\ \mathrm{B} & \mathrm{B}' & \mathrm{B}'' \end{vmatrix} = \begin{vmatrix} \xi & \eta & \zeta \\ \mathrm{A} & \mathrm{A}' & \mathrm{A}'' \\ \mathrm{B} & \mathrm{B}' & \mathrm{B}'' \end{vmatrix} = \mathrm{D}. \qquad \ldots \quad (27)$$

From (24) it is clear that C, C′, C″ are the trilinear coordinates of the point which is the origin of coordinates in the cartesian system. Hence also

$$\begin{vmatrix} \mathrm{A} & \mathrm{A}' & \mathrm{A}'' \\ \mathrm{B} & \mathrm{B}' & \mathrm{B}'' \\ \mathrm{C} & \mathrm{C}' & \mathrm{C}'' \end{vmatrix} = \mathrm{D}. \qquad \ldots \quad (28)$$

7

In what follows we shall make frequent use of the following determinant identity :[1]

$$\begin{vmatrix} f & g & h \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix} \begin{vmatrix} a & a' & a'' \\ l & m & n \\ p & q & r \end{vmatrix} + \begin{vmatrix} a & a' & a'' \\ f & g & h \\ c & c' & c'' \end{vmatrix} \begin{vmatrix} b & b' & b'' \\ l & m & n \\ p & q & r \end{vmatrix} + \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ f & g & h \end{vmatrix} \begin{vmatrix} c & c' & c'' \\ l & m & n \\ p & q & r \end{vmatrix}$$

[1] The identities (29) and (30) are but particular cases of the following more general Determinantal Identity :

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r1} & a_{r2} & \cdot & a_{rn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

$$\equiv \sum_{m=1}^{m=n} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{11} & b_{12} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \begin{vmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

where the summation consists of n products which are derived from the product of the two determinants on the left-hand side by interchanging the first row of the second detetminant with the successive rows of the first determinant.

$$\equiv\begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix}\begin{vmatrix} f & g & h \\ l & m & n \\ p & q & r \end{vmatrix} \quad \dots \quad (29)$$

which is easily proved.

If in (29) we replace c, c', c'' by l, m, n respectively we get

$$\begin{vmatrix} f & g & h \\ b & b' & b'' \\ l & m & n \end{vmatrix}\begin{vmatrix} a & a' & a'' \\ l & m & n \\ p & q & r \end{vmatrix}+\begin{vmatrix} a & a' & a'' \\ f & g & h \\ l & m & n \end{vmatrix}\begin{vmatrix} b & b' & b'' \\ l & m & n \\ p & q & r \end{vmatrix}\equiv\begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ l & m & n \end{vmatrix}\begin{vmatrix} f & g & h \\ l & m & n \\ p & q & r \end{vmatrix}$$

which rewritten is

$$\begin{vmatrix} l & m & n \\ a & a' & a'' \\ f & g & h \end{vmatrix}\begin{vmatrix} l & m & n \\ b & b' & b'' \\ p & q & r \end{vmatrix}-\begin{vmatrix} l & m & n \\ a & a' & a'' \\ p & q & r \end{vmatrix}\begin{vmatrix} l & m & n \\ b & b' & b'' \\ f & g & h \end{vmatrix}$$

$$\equiv\begin{vmatrix} l & m & n \\ a & a' & a'' \\ b & b' & b'' \end{vmatrix}\begin{vmatrix} l & m & n \\ f & g & h \\ p & q & r \end{vmatrix} \quad \dots \quad (30)$$

The identity in this latter form will be most useful.

8

From equations (24) we obtain

$$X=\frac{1}{D}\begin{vmatrix} \xi & \eta & \zeta \\ B & B' & B'' \\ C & C' & C'' \end{vmatrix}, \quad \dots \quad (31)$$

$$Y=\frac{1}{D}\begin{vmatrix} \xi & \eta & \zeta \\ C & C' & C'' \\ A & A' & A'' \end{vmatrix}. \quad \dots \quad (32)$$

From (31) by differentiation we get

$$X'=\frac{1}{D^2}\left\{\begin{vmatrix} \xi' & \eta' & \zeta' \\ B & B' & B'' \\ C & C' & C'' \end{vmatrix}\begin{vmatrix} \xi & \eta & \zeta \\ A & A' & A'' \\ B & B' & B'' \end{vmatrix}-\begin{vmatrix} \xi & \eta & \zeta \\ B & B' & B'' \\ C & C' & C'' \end{vmatrix}\begin{vmatrix} \xi' & \eta' & \zeta' \\ A & A' & A'' \\ B & B' & B'' \end{vmatrix}\right\},$$

$$=\frac{1}{D^2}\begin{vmatrix} A & A' & A'' \\ B & B' & B'' \\ C & C' & C'' \end{vmatrix}\begin{vmatrix} B & B' & B'' \\ \xi & \eta & \zeta \\ \xi' & \eta' & \zeta' \end{vmatrix} \text{ by (30),}$$

$$=\frac{1}{D}\begin{vmatrix} B & B' & B'' \\ \xi & \eta & \zeta \\ \xi' & \eta' & \zeta' \end{vmatrix} \text{ by (28).} \qquad \ldots \quad (33)$$

Similarly,

$$Y'=-\frac{1}{D}\begin{vmatrix} A & A' & A'' \\ \xi & \eta & \zeta \\ \xi' & \eta' & \zeta' \end{vmatrix}. \qquad \ldots \quad (34)$$

Again differentiating X' and Y' we have

$$X''=\frac{1}{D}\begin{vmatrix} B & B' & B'' \\ \xi & \eta & \zeta \\ \xi'' & \eta'' & \zeta'' \end{vmatrix}, \qquad \ldots \quad (35)$$

$$Y''=-\frac{1}{D}\begin{vmatrix} A & A' & A'' \\ \xi & \eta & \zeta \\ \xi'' & \eta'' & \zeta'' \end{vmatrix}. \qquad \ldots \quad (36)$$

From (35) by further differentiation

$$X'''=\frac{1}{D}\left\{\begin{vmatrix} B & B' & B'' \\ \xi' & \eta' & \zeta' \\ \xi'' & \eta'' & \zeta'' \end{vmatrix}+\begin{vmatrix} B & B' & B'' \\ \xi & \eta & \zeta \\ \xi''' & \eta''' & \zeta''' \end{vmatrix}\right\}$$

Now since

$$l\xi+m\eta+n\zeta=1,$$

we have

$$l\xi'+m\eta'+n\zeta'=0,$$

$$l\xi''+m\eta''+n\zeta''=0.$$

$$\therefore \frac{l}{\eta'\zeta''-\eta''\zeta'}=\frac{m}{\zeta'\xi''-\zeta''\xi'}=\frac{n}{\xi'\eta''-\xi''\eta'}=\frac{1}{k} \text{ say.} \qquad \ldots \quad (37)$$

Again

$$A'B''-A''B'=\mu\nu\{\cos\phi\sin\psi-\cos\psi\sin\phi\}=\mu\nu\sin\widehat{bc}$$

$$\left.\begin{aligned} &=\frac{a\mu\nu}{2\rho}=\lambda\mu\nu\frac{\Delta}{\rho}l,\\ A''B-AB''&=\frac{b\nu\lambda}{2\rho}=\lambda\mu\nu\frac{\Delta}{\rho}m,\\ AB'-A'B&=\frac{c\lambda\mu}{2\rho}=\lambda\mu\nu\frac{\Delta}{\rho}n,\end{aligned}\right\} \quad \ldots \quad (38)$$

The determinant

$$\begin{vmatrix} B & B' & B'' \\ \xi' & \eta' & \zeta' \\ \xi'' & \eta'' & \zeta'' \end{vmatrix}=B(\eta'\zeta''-\eta''\zeta')+B'(\zeta'\xi''-\zeta''\xi')+B''(\xi'\eta''-\xi''\eta')$$

$$=\frac{k}{\lambda\mu\nu}\cdot\frac{\rho}{\Delta}\{B(A'B''-A''B')+B'(A''B-AB'')$$

$$+B''(AB'-A'B)\}\text{ from (37) and (38),}$$

$$=0$$

Hence

$$X'''=\frac{1}{D}\begin{vmatrix} B & B' & B'' \\ \xi & \eta & \zeta \\ \xi''' & \eta''' & \zeta''' \end{vmatrix} \quad \ldots \quad (39)$$

Similarly

$$Y'''=-\frac{1}{D}\begin{vmatrix} A & A' & A'' \\ \xi & \eta & \zeta \\ \xi''' & \eta''' & \zeta''' \end{vmatrix} \quad \ldots \quad (40)$$

Proceeding as above we can show that

$$X^{(n)}=\frac{1}{D}\begin{vmatrix} B & B' & B'' \\ \xi & \eta & \zeta \\ \xi^{(n)} & \eta^{(n)} & \zeta^{(n)} \end{vmatrix} \quad \ldots \quad (41)$$

$$Y^{(n)}=-\frac{1}{D}\begin{vmatrix} A & A' & A'' \\ \xi & \eta & \zeta \\ \xi^{(n)} & \eta^{(n)} & \zeta^{(n)} \end{vmatrix}, \qquad \ldots \quad (42)$$

We shall now make use of these transformation formulae to find the values of the functions P, Q, R, S, T, Q_1, R$'$, S$'$ which are defined below.

We have

$$x^{(m)}y^{(n)}-x^{(n)}y^{(m)}$$

$$=\frac{1}{D^2}\left\{\begin{vmatrix} \alpha & \beta & \gamma \\ A & A' & A'' \\ \alpha^{(m)} & \beta^{(m)} & \gamma^{(m)} \end{vmatrix}\begin{vmatrix} \alpha & \beta & \gamma \\ B & B' & B'' \\ \alpha^{(n)} & \beta^{(n)} & \gamma^{(n)} \end{vmatrix}\right.$$

$$\left.-\begin{vmatrix} \alpha & \beta & \gamma \\ A & A' & A'' \\ \alpha^{(n)} & \beta^{(n)} & \gamma^{(n)} \end{vmatrix}\begin{vmatrix} \alpha & \beta & \gamma \\ B & B' & B'' \\ \alpha^{(m)} & \beta^{(m)} & \gamma^{(m)} \end{vmatrix}\right\},$$

$$=\frac{1}{D^2}\begin{vmatrix} \alpha & \beta & \gamma \\ A & A' & A'' \\ B & B' & B'' \end{vmatrix}\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^{(m)} & \beta^{(m)} & \gamma^{(m)} \\ \alpha^{(n)} & \beta^{(n)} & \gamma^{(n)} \end{vmatrix} \text{ by (30)},$$

$$=\frac{\Delta_{mn}}{D}.$$

Hence

$$\left.\begin{aligned}
Q&=x'y''-x''y'=\frac{\Delta_{12}}{D},\\
R&=x'y'''-x'''y'=\frac{\Delta_{13}}{D},\\
S&=x'y^{iv}-x^{iv}y'=\frac{\Delta_{14}}{D},\\
T&=x'y^{v}-x^{v}y'=\frac{\Delta_{15}}{D},\\
R'&=x''y'''-x'''y''=\frac{\Delta_{23}}{D},\\
S'&=x''y^{iv}-x^{iv}y''=\frac{\Delta_{24}}{D},
\end{aligned}\right\} \quad \ldots \quad (43)$$

$$P=x'^2+y'^2=\frac{1}{D^2}\left\{\begin{vmatrix} A & A' & A'' \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}^2+\begin{vmatrix} B & B' & B'' \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}^2\right\} \quad \ldots \quad (44)$$

$$Q_1=x'x''+y'y''=\frac{1}{D^2}\left\{\begin{vmatrix} A & A' & A'' \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}\begin{vmatrix} A & A' & A'' \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}\right.$$

$$\left.+\begin{vmatrix} B & B' & B'' \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}\begin{vmatrix} B & B' & B'' \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}\right\} \quad \ldots \quad (45)$$

Now the expression

$$(Y-y)x'-(X-x)y'$$

or

$$Yx'-Xy'-x'y+xy'$$

becomes on transformation

$$=\frac{1}{D^2}\left\{\begin{vmatrix}\xi & \eta & \zeta\\ C & C' & C''\\ A & A' & A''\end{vmatrix}\begin{vmatrix}B & B' & B''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}+\begin{vmatrix}\xi & \eta & \zeta\\ B & B' & B''\\ C & C' & C''\end{vmatrix}\begin{vmatrix}A & A' & A''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\right.$$

$$\left.+\begin{vmatrix}A & A' & A''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\begin{vmatrix}B & B' & B''\\ \alpha & \beta & \gamma\\ C' & C' & C'\end{vmatrix}-\begin{vmatrix}A & A' & A''\\ \alpha & \beta & \gamma\\ C & C' & C''\end{vmatrix}\begin{vmatrix}B & B' & B''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\right\},$$

$$=\frac{1}{D^2}\begin{vmatrix}\xi & \eta & \zeta\\ B & B' & B''\\ C & C' & C''\end{vmatrix}\begin{vmatrix}A & A' & A''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}+\begin{vmatrix}\xi & \eta & \zeta\\ C & C' & C''\\ A & A' & A''\end{vmatrix}\begin{vmatrix}B & B' & B''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}$$

$$\left.+\begin{vmatrix}\alpha & \beta & \gamma\\ A & A' & A''\\ B & B' & B''\end{vmatrix}\begin{vmatrix}C & C' & C''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\right\}\text{ by (30),}$$

$$=\frac{1}{D^2}\left\{\begin{vmatrix}\xi & \eta & \zeta\\ B & B' & B''\\ C & C' & C''\end{vmatrix}\begin{vmatrix}A & A' & A''\\ & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}+\begin{vmatrix}A & A' & A''\\ \xi & \eta & \zeta\\ C & C' & C''\end{vmatrix}\begin{vmatrix}B & B' & B''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\right.$$

$$\left.+\begin{vmatrix}A & A' & A''\\ B & B' & B''\\ \xi & \eta & \zeta\end{vmatrix}\begin{vmatrix}C & C' & C''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\right.\text{ by (27),}$$

$$=\frac{1}{D^2}\left\{\begin{vmatrix}A & A' & A''\\ B & B' & B''\\ C & C' & C''\end{vmatrix}\begin{vmatrix}\xi & \eta & \zeta\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\right.\text{ by (29),}$$

$$=\frac{1}{D}\begin{vmatrix}\xi & \eta & \zeta\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\text{ by (28).} \qquad \ldots \quad (46)$$

Similarly the expression

$$(Y-y)x''-(X-x)y''$$

or $$Yx''-Xy''-x''y+xy''$$

reduces on transformation to

$$\frac{1}{D}\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}. \qquad \ldots \quad (47)$$

9

The equation to the osculating parabola at a point x, y on the curve is

$$\{(Y-y)(3Qx''-Rx')-(X-x)(3Qy''-Ry')\}^2$$

$$=18Q^3\{(Y-y)x'-(X-x)y'\},$$

[See S. Mukhopadhyaya—A General Theory of Osculating Conics, Second paper : *Journal and Proceedings, Asiatic Society of Bengal (New Series) Vol. IV, No. 10, 1908.*]

or, $$\{3Q[(Y-y)x''-(X-x)y'']-R[(Y-y)x'-(X-x)y']\}^2$$

$$=18Q^3\{(Y-y)x'-(X-x)y'\};$$

this on being transformed is

$$3\Delta_{12}\left\{\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}-\Delta_{13}\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}\right\}^2=18\Delta_{12}^3\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}$$

or, $$3\Delta_{12}\left\{\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}-\Delta_{13}\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}\right\}^2$$

$$=18\Delta_{12}^2\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}\begin{vmatrix} \xi & \eta & \zeta \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} \text{ by (14)} \qquad \ldots \quad (48)$$

or $$(\Delta_{13}U-3\Delta_{12}V)^2=18\Delta_{12}^2UW.$$

Again the equation to the osculating conic at a point (x,y) on the curve is

$$\{(Y-y)(3Qx''-Rx')-(X-x)(3Qy''-Ry')\}^2$$
$$+(3QS-5R^2+12QR')\{(Y-y)x'-(X-x)y'\}^2$$
$$=18Q^3\{(Y-y)x'-(X-x)y'\};$$

[See, A General Theory of Osculating Conics, *loc. cit.* (52).]

or,
$$\{3Q[(Y-y)x''-(X-x)y'']-R[(Y-y)x'-(X-x)y']\}^2$$
$$+(3QS-5R^2+12QR')\{(Y-y)x'-(X-x)y'\}^2$$
$$=18Q^3\{(Y-y)x'-(X-x)y'\}.$$

This transforms into

$$3\Delta_{12}\left\{\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}-\Delta_{13}\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}\right\}^2$$
$$+(12\Delta_{12}\Delta_{23}-5\Delta_{13}^2+3\Delta_{12}\Delta_{14})\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}^2$$
$$=18\Delta_{12}^4\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix},$$

or
$$3\Delta_{12}\left\{\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}-\Delta_{13}\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}\right\}^2$$
$$+(12\Delta_{12}\Delta_{23}-5\Delta_{13}^2+3\Delta_{12}\Delta_{14})\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}^2$$
$$=18\Delta_{12}^3\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}\begin{vmatrix} \xi & \eta & \zeta \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} \text{ by (14), (49)}$$

or $$(\Delta_{12}^{2}U-3\Delta_{12}V)^2+\sigma U^2=18\Delta_{12}^2 UW,$$

which is the same as equation (3).

Finally if the osculating conic has a six-pointic contact at α, β, γ we must have

$$40R^3-45QRS+9Q^2T-90QRR'+45Q^2S'=0,$$

[See, A General Theory of Osculating Conics, *loc. cit.*] which transforms into relation (18) of § 4.

This paper was written at the suggestion and under the guidance of Prof. S. Mukhopadhyaya to whom my best thanks are due.

9

On some Laws of Central Force.

Part I.

BY

N. M. Basu, M.Sc.

1. The law of Central force under which a particle can describe a a conic, whatever may be the conditions of projection, has formed the subject of investigation by mathematicians from very early times. Among the contributors to the subject may be mentioned the names of Newton,[1] Sir W. R. Hamilton,[2] Villarceau,[3] Darboux,[4] Halphen, Glaisher,[5] Hirayama,[6] Appell[7] and others. These mathematicians confined themselves to the consideration of the law of force as being a function of the position of the particle relative to the centre of force.

2. The problem of determining the law of force when it depends on the velocity as well as on the position of the moving particle appears to have been first solved by M. Paul J. Suchar[8] who has shown that, besides the two well-known laws of force depending on the position, there are six and only six other laws depending on the velocity as well. These six laws are :—

(*i*) $\mu(a\dot{x}_1+b\dot{y}_1+c)^3 r_1$, (*ii*) $\mu(a\dot{x}_1^2+2b\dot{x}_1\dot{y}_1+c\dot{y}_1^2)^{\frac{3}{2}} r_1$,

(*iii*) $\mu\dot{y}_1^3(ax_1+by_1+c)^{-3} r_1$ (*iv*) $\mu\dot{y}_1^3(ay_1^2+2by_1+c)^{-\frac{3}{2}} r_1$,

v) $\mu y_1^{-3}(a\dot{x}_1+b\dot{y}_1+c)^3 r_1$, (*vi*) $\mu y_1^{-3}(a\dot{y}_1^2+2b\dot{y}_1+c)^{\frac{3}{2}} r_1$,

and the laws of force previously known are

(*vii*) $\mu(ax_1+by_1+c)^{-3} r_1$, (*viii*) $\mu(ax_1^2+2bx_1y_1+cy_1^2)^{-\frac{3}{2}} r_1$,

[1] First book of the Principia.

[2] Proceedings of the Irish Academy, Vol. 3, 1846.

[3] Connaissance des Temps, 1852.

[4] Comptes Rendus, Vol. 84, 1877.

[5] Monthly Notices of the Royal Astronomical Society, Vol. 39, 1878.

[6] Gould's Astronomical Journal, 1889.

[7] American Journal of Mathematics, Vol. 13, 1891.

[8] Nouvelles Annales de Mathematiques, Vol. 6, Series, 4, 1906.

where x_1, y_1, are the co-ordinates and r_1 the distance from the origin of the moving particle at time t_1, μ, a, b, c are arbitrary constants and $\dot{x}_1, \dot{y}_1$ stand for $\frac{dx_1}{dt_1}$ and $\frac{dy_1}{dt_1}$ respectively.

3. In obtaining these laws Suchar has made use of a very important theorem established by himself.[1] Using the transformation,

$$\dot{x}=\frac{dx}{dt}=x', \quad \dot{y}=\frac{dy}{dt}=y'. \quad \frac{dt'}{dt}=\mathrm{F}(x, y, \dot{x}, \dot{y})$$

where (x,y) are the co-ordinates, at time t, of a particle moving under a central force at the origin whose acceleration per unit mass is $r\mathrm{F}(x,y,\dot{x},\dot{y})$, where $r=(x^2+y^2)^{\frac{1}{2}}$, we obtain

$$\dot{x}' = \frac{dx'}{dt'} = x, \quad \dot{y}' = \frac{dy'}{dt'} = y,$$

whence we further obtain

$$\ddot{x}'=\frac{d^2x'}{dt'^2} = \frac{x'}{\mathrm{F}(x,y,\dot{x},\dot{y})} = \frac{x'}{\mathrm{F}(\dot{x}',\dot{y}',x',y')}$$

and

$$\ddot{y}'=\frac{d^2y'}{dt'^2} = \frac{y'}{\mathrm{F}(x,y,\dot{x},\dot{y})} = \frac{y'}{\mathrm{F}(\dot{x}',\dot{y}',x',y')}.$$

It thus follows that if (x',y') denotes the position of a second moving particle at the time t', this particle moves under a central force at the origin whose acceleration per unit mass is $r'/\mathrm{F}(\dot{x}',\dot{y}',x',y')$, r' being its distance from the origin. But the transformation formulae shew that each particle describes the hodograph of the other. Remembering that the hodograph of a conic described under a central force is itself a conic it is thus proved that *if a conic be described under the central force* $r\mathrm{F}(x, y, \dot{x}, \dot{y})$, *a conic will also be described under the force* $r/\mathrm{F}(\dot{x}, \dot{y}, x, y)$. (We shall hereafter refer to this as Suchar's Theorem.)

4. The method employed by Suchar in obtaining the laws (*i*) to (*vi*) is rather tentative The object of the present paper is to give a more general and rigorous, though less simple, method by which I have been able to obtain all the laws excepting (*i*) and (*ii*) without using Suchar's Theorem, and by which, I believe, the first two laws can also

[1] Bull. de la Soc. des Sciences, t XXXIII, Comptes Rendus, Vol. 135, p. 679.

be obtained. These two laws can however be easily obtained from the well-known laws (*vii*) and (*viii*) by the application of Suchar's theorem.

5. Let (x_1, y_1) be the co-ordinates and r_1 the distance from the origin of a particle moving under the central force F_1, at the time t_1. Its equations of motion are

$$\frac{d^2x_1}{dt_1^2}=F_1\frac{x_1}{r_1},\qquad \frac{d^2y_1}{dt_1^2}=F_1\frac{y_1}{r_1}$$

whence

$$x_1\frac{dy_1}{dt_1}-y_1\frac{dx_1}{dt_1}=\text{a constant}=h\text{(say)}$$

Making the homographic transformation

$$\frac{x_1}{y_1}=x,\quad y_1=\frac{1}{y},\quad \frac{dt_1}{y_1^2}=dt,$$

we have

$$\frac{dx}{dt}=-h,\ \frac{dy}{dt}=-\frac{dy_1}{dt_1}$$

whence

$$\frac{d^2x}{dt^2}=0,\quad \frac{d^2y}{dt^2}=-\frac{d^2y_1}{dt_1^2}\cdot\frac{dt_1}{dt}=-F_1\frac{y_1^3}{r_1}.$$

Thus the equations of motion of a particle, whose co-ordinates are (x,y) at the time t, are

$$\frac{d^2x}{dt^2}=0,\quad \frac{d^2y}{dt^2}=Y \qquad \cdots\ (5\cdot1)$$

where
$$Y=-F_1\frac{y_1^3}{r_1} \qquad \cdots\ (5\cdot2)$$

From the nature of the transformation it follows that if the curve described by the first particle be a conic, that described by the second will also be a conic. The determination of F_1 thus reduces to the determination of a force Y acting parrallel to a fixed direction under which a particle can describe a conic.

6. Since the equations of motion of (x,y) are $(5\cdot1)$, the differential equation of its trajectory is

$$\frac{d^2y}{dx^2}=\frac{Y}{h^2},$$

where $$\frac{dv}{dt}=-h.$$

Now the differential equation of all conics is

$$\frac{d^3}{dx^3}\left[\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right]=0.$$

Thus Y must satisfy the equation

$$\frac{d^3}{dx^3}\left[(Y)^{-\frac{2}{3}}\right]=0.$$

Since the initial conditions are perfectly arbitrary, Y must be such as to satisfy this equation for all values of x, y, h and $\frac{dy}{dx}$.

7. Let us assume Y to be given by

$$Y=[\phi(x,y,y')]^{-\frac{3}{2}} \qquad \ldots \quad (7\cdot 1)$$

where y' stands for $\frac{dy}{dx}$.

Then ϕ must satisfy the equation

$$\frac{d^3}{dx^3}\,[\phi(x,y,y')]=0.$$

Performing the differentiation and remembering that

$$\frac{d^2y}{dx^2}=\frac{Y}{h^2}=\frac{\phi^{-\frac{3}{2}}}{h^2},$$

the above equation becomes

$$\frac{\partial^3\phi}{\partial x^3}+3y'\frac{\partial^3\phi}{\partial x^2\partial y}+3y'^2\frac{\partial^3\phi}{\partial x\partial y^2}+y'^3\frac{\partial^3\phi}{\partial y^3}$$

$$+\frac{3\phi^{-\frac{7}{2}}}{4h^2}\Bigg[4\phi^2\left\{\frac{\partial^3\phi}{\partial x^2\partial y'}+\frac{\partial^2\phi}{\partial x\partial y}+y'\left(2\frac{\partial^3\phi}{\partial x\partial y\partial y'}+\frac{\partial^2\phi}{\partial y^2}\right)\right.$$

$$\left.+y'^2\frac{\partial^3\phi}{\partial y^2\partial y'}\right\}-2\phi\left\{\frac{\partial\phi}{\partial x}\cdot\frac{\partial\phi}{\partial y}+3\frac{\partial\phi}{\partial x}\cdot\frac{\partial}{\partial x\partial y'}\right.$$

$$\left.+\frac{\partial^2\phi}{\partial x^2}\cdot\frac{\partial\phi}{\partial y'}\right\}-2y'\phi\left\{\left(\frac{\partial\phi}{\partial y}\right)^2+3\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial x\partial y'}+3\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y\partial y'}\right.$$

$$+2\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial x\partial y}\Big\}+5\frac{\partial\phi}{\partial y'}\Big\{\Big(\frac{\partial\phi}{\partial x}\Big)^2+2y'\frac{\partial\phi}{\partial x}\cdot\frac{\partial\phi}{\partial y}\Big\}$$

$$+y'^2\Big\{5\frac{\partial\phi}{\partial y'}\Big(\frac{\partial\phi}{\partial y}\Big)^2-2\phi\Big(3\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial y\partial y'}+\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial y^2}\Big)\Big\}\Big]$$

$$+\frac{3\phi^{-5}}{4h^4}\Big[4\phi^2\Big(\frac{\partial^3\phi}{\partial\cdot\partial y'^2}+\frac{\partial^2\phi}{\partial y\partial y'}\Big)-2\phi\Big(2\frac{\partial\phi}{\partial y}\cdot\frac{\partial\phi}{\partial y'}$$

$$+5\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial x\partial y'}+3\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y'^2}\Big)+13\Big(\frac{\partial\phi}{\partial y'}\Big)^2\Big(\frac{\partial\phi}{\partial x}+y'\frac{\partial\phi}{\partial y}\Big)$$

$$+y'\Big\{4\phi^2\frac{\partial^3\phi}{\partial y\partial y'^2}-2\phi\Big(5\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial y\partial y'}+3\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial y'^2}\Big)\Big\}\Big]$$

$$+\frac{\phi^{-\frac{11}{2}}}{h^6}\Big[\phi^2\frac{\partial^3\phi}{\partial y'^3}-6\phi\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial y'^2}+6\Big(\frac{\partial\phi}{\partial y'}\Big)^3\Big]=0.$$

8. Since the above equation is to be true for all values of h, ϕ must satisfy the following four differential equations:

$$\frac{\partial^3\phi}{\partial x^3}+3y'\frac{\partial^3\phi}{\partial x^2\partial y}+3y'^2\frac{\partial^3\phi}{\partial x\partial y^2}+y'^3\frac{\partial^3\phi}{\partial y^3}=0 \quad\dots\quad (I)$$

$$4\phi^2\Big\{\frac{\partial^3\phi}{\partial x^2\partial y'}+\frac{\partial^2\phi}{\partial x\partial y}+y'\Big(2\frac{\partial^3\phi}{\partial x\partial y\partial y'}+\frac{\partial^2\phi}{\partial y^2}\Big)+y'^2\frac{\partial^3\phi}{\partial y^2\partial y'}\Big\}$$

$$-2\phi\Big[\frac{\partial\phi}{\partial x}\cdot\frac{\partial\phi}{\partial y}+3\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x\partial y'}+\frac{\partial^2\phi}{\partial x^2}\cdot\frac{\partial\phi}{\partial y'}$$

$$+y'\Big\{\Big(\frac{\partial\phi}{\partial y}\Big)^2+3\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial x\partial y'}+3\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y\partial y'}$$

$$+2\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial x\partial y}\Big\}\Big]+5\frac{\partial\phi}{\partial y'}\Big\{\Big(\frac{\partial\phi}{\partial x}\Big)^2+2y'\frac{\partial\phi}{\partial x}\cdot\frac{\partial\phi}{\partial y}\Big\}$$

$$+y'^2\Big\{5\frac{\partial\phi}{\partial y'}\Big(\frac{\partial\phi}{\partial y}\Big)^2-2\phi\Big(3\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial y\partial y'}$$

$$+\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial y^2}\Big)\Big\}=0. \quad\dots\quad (II)$$

$$4\phi^2\left(\frac{\partial^3\phi}{\partial x\,\partial y'^2}+\frac{\partial^2\phi}{\partial y\partial y'}\right)-2\phi\left(2\frac{\partial\phi}{\partial y}\cdot\frac{\partial\phi}{\partial y'}+5\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial x\,\partial y'}\right.$$

$$\left.+3\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y'^2}\right)+y'\left\{4\phi^2\frac{\partial^3\phi}{\partial y\,\partial y'^2}-2\phi\left(5\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial y\,\partial y'}\right.\right.$$

$$\left.\left.+3\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial y'^2}\right)\right\}+13\left(\frac{\partial\phi}{\partial y'}\right)^2\left(\frac{\partial\phi}{\partial x}+y'\frac{\partial\phi}{\partial y}\right)=0. \quad \dots \quad \text{(III)}$$

$$\phi^2\frac{\partial^3\phi}{\partial y'^3}-6\phi\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial y'^2}+6\left(\frac{\partial\phi}{\partial y'}\right)^3=0. \quad \dots \quad \text{(IV)}$$

9. In order to solve these four equations simultaneously, consider at first the equation (IV).

Putting $\phi=\dfrac{1}{z}$, this reduces to

$$\frac{d^3z}{dy'^3}=0.$$

Hence $$z=\frac{1}{\phi}=f_1(x,y)y'^2+f_2(x,y)y'+f_3(x,y) \quad \dots \quad (9\cdot1)$$

where f_1, f_2 and f_3 are some unknown functions to be determined from the equations (I), (II) and (III).

The laws (VII) and (VIII) can be easily deduced, as has already been shewn by Appell,[1] by assuming ϕ to be independent of y' in which case the differential equations become very much simplified and can be completely solved. In the present case the general solution of the equations seems very difficult and we shall try to obtain some particular solutions only.

10. Let us assume in the first place that

$$f_2=0,\ f_3=0 \text{ and } f_1=\frac{1}{f(x,y)}, \text{ so that } \phi=\frac{f(x,y)}{y'^2}.$$

Substituting this value in the equations (I), (II) and (III) and remembering that they must be satisfied for all values of y', we obtain the following six equations

$$\frac{\partial^3 f}{\partial x^3}=0,\quad \frac{\partial^3 f}{\partial x^2\partial y}=0,\quad \frac{\partial^3 f}{\partial x\,\partial y^2}=0,\quad \frac{\partial^3 f}{\partial y^3}=0,$$

$$2f\frac{\partial^2 f}{\partial x^2}-\left(\frac{\partial f}{\partial x}\right)^2=0,\quad 2f\frac{\partial^2 f}{\partial x\,\partial y}-\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}=0.$$

[1] loc. cit

The first four of these equations shew that f must be at most a quadratic function of x and y, and the last two equations shew that f must have one of the forms

$$f=(ax+by+c)^2$$

or

$$f=(ay^2+2by+c).$$

Since

$$y'=\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{1}{h}\frac{dy_1}{dt_1}=\frac{1}{h}\dot{y}_1,$$

and

$$x=\frac{x_1}{y_1}, \quad y=\frac{1}{y_1},$$

we have

$$\phi=\frac{(ax+by'+c)^2}{y'^3}=\frac{h^3(ax_1+cy_1+b)^2}{y_1^3\dot{y}_1^3}$$

or

$$\phi=\frac{ay'^2+2by'+c}{y'^3}=\frac{h^3(c\dot{y}_1^2+2b\dot{y}_1+a)}{y_1^3\dot{y}_1^3}.$$

Hence, from $(5\cdot2)$ and $(7\cdot1)$.

$$F_1=h^2y_1^3(ax_1+cy_1+b)^{-3}r_1$$

or

$$F_1=h^2y_1^3(cy_1^2+2by_1+a)^{-\frac{3}{2}}r_1.$$

These are evidently the forms (*iii*) and (*iv*) of Suchar.

11. Let us now attempt to find a more general solution. Substituting the value of ϕ given by $(9\cdot1)$ in the equation (I), we obtain, on equating to zero the coefficient of the highest power of y' and the term independent of y', the following two equations

$$6\left(\frac{\partial f_1}{\partial y}\right)^3-6f_1\frac{\partial f_1}{\partial y}\cdot\frac{\partial^2 f_1}{\partial y^2}+f_1^2\frac{\partial^3 f_1}{\partial y^3}=0 \qquad \ldots \quad (11\cdot1)$$

$$6\left(\frac{\partial f_3}{\partial x}\right)^3-6f_3\frac{\partial f_3}{\partial x}\cdot\frac{\partial^2 f_3}{\partial x^2}+f_3^2\frac{\partial^3 f_3}{\partial x^3}=0 \qquad \ldots \quad (11\cdot2)$$

These equations being of the same form as (IV), their general solutions are

$$f_1(x,y)=\{\theta_1(x)y^2+\theta_2(x)y+\theta_3(x)\}^{-1} \qquad \ldots \quad (11\cdot3)$$

$$f_3(x,y)=\{\psi_1(y)x^2+\psi_2(y)x+\psi_3(y)\}^{-1} \qquad \ldots \quad (11\cdot4)$$

where θ_1, θ_2, θ_3, ψ_1, ψ_2 and ψ_3 are unknown functions to be determined from the remaining equations.

As the adoption of these general solutions is bound to lead to complications, we shall assume for the present that f_1 is a function of x only and f_3 that of y only, which is tantamount to the assumption

$$\theta_1 = \theta_2 = \psi_1 = \psi_2 = 0.$$

Making this assumption and equating to zero the coefficients of the other powers of y', we obtain the following eight equations

$$\frac{\partial^3 f_2}{\partial x^3} = 0 \qquad \ldots \quad (11\cdot 41)$$

$$\frac{\partial^3 f_2}{\partial y^3} = 0 \qquad \ldots \quad (11\cdot 42)$$

$$6\,\frac{\partial^2 f_2}{\partial x^2}\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_3}{\partial y}\right) - f_3\left(\frac{\partial^3 f_1}{\partial x^3} + 3\,\frac{\partial^3 f_2}{\partial x^2 \partial y}\right) = 0 \qquad \ldots \quad (11\cdot 43)$$

$$6\,\frac{\partial^2 f_2}{\partial y^2}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) - f_1\left(\frac{\partial^3 f_3}{\partial y^3} + 3\,\frac{\partial^3 f_2}{\partial x \partial y^2}\right) = 0 \qquad \ldots \quad (11\cdot 44)$$

$$6\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_3}{\partial y}\right)\left[f_2\,\frac{\partial^2 f_2}{\partial x^2} + f_3\left(\frac{\partial^2 f_1}{\partial x^2} + 2\,\frac{\partial^2 f_2}{\partial x \partial y} + \frac{\partial^2 f_3}{\partial y^2}\right)\right.$$

$$\left. - \left(\frac{\partial f_2}{\partial x} + \frac{\partial f_3}{\partial y}\right)^2\right] + 6 f_3\,\frac{\partial^2 f_2}{\partial x^2}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right)$$

$$-2 f_2 f_3\left(\frac{\partial^3 f_1}{\partial x^3} + 3\,\frac{\partial^3 f_2}{\partial x^2 \partial y}\right) - f_3^2\left(\frac{\partial^3 f_3}{\partial y^3} + 3\,\frac{\partial^3 f_2}{\partial x \partial y^2}\right) = 0 \quad (11\cdot 45)$$

$$6\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right)\left[f_2\,\frac{\partial^2 f_2}{\partial y^2} + f_1\left(\frac{\partial^2 f_1}{\partial x^2} + 2\,\frac{\partial^2 f_2}{\partial x \partial y} + \frac{\partial^2 f_3}{\partial y^2}\right)\right.$$

$$\left. - \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right)^2\right] + 6 f_1\,\frac{\partial^2 f_2}{\partial y^2}\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_3}{\partial y}\right)$$

$$-2 f_1 f_2\left(\frac{\partial^3 f_3}{\partial y^3} + 3\,\frac{\partial^3 f_2}{\partial x \partial y^2}\right) - f_1^2\left(\frac{\partial^3 f_1}{\partial x^3} + 3\,\frac{\partial^3 f_2}{\partial x^2 \partial y}\right) = 0 \quad (11\cdot 46)$$

$$6\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right)\left[f_2\,\frac{\partial^2 f_2}{\partial x^2} + f_3\left(\frac{\partial^2 f_1}{\partial x^2} + 2\,\frac{\partial^2 f_2}{\partial x \partial y} + \frac{\partial^2 f_3}{\partial y^2}\right)\right.$$

$$\left. -3\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_3}{\partial y}\right)^2\right] + 6\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_3}{\partial y}\right)\left[f_1\,\frac{\partial^2 f_2}{\partial x^2}\right.$$

$$+f_2\left(\frac{\partial^2 f_1}{\partial x^2}+2\frac{\partial^2 f_2}{\partial x\partial y}+\frac{\partial^2 f_3}{\partial y^2}\right)+f_3\frac{\partial^2 f_2}{\partial y^2}\Bigg]$$

$$-2f_2f_3\left(\frac{\partial^3 f_3}{\partial y^3}+3\frac{\partial^3 f_2}{\partial x\partial y^2}\right)$$

$$-(f_2^2+2f_1f_3)\left(\frac{\partial^3 f_1}{\partial x^3}+3\frac{\partial^3 f_2}{\partial x^2\partial y}\right)=0 \qquad \dots \quad (11\cdot 47)$$

$$6\left(\frac{\partial f_2}{\partial x}+\frac{\partial f_3}{\partial y}\right)\Bigg[f_2\frac{\partial^2 f_2}{\partial y^2}+f_1\left(\frac{\partial^2 f_1}{\partial x^2}+2\frac{\partial^2 f_2}{\partial x\partial y}+\frac{\partial^2 f_3}{\partial y^2}\right)$$

$$-3\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}\right)^2\Bigg]+6\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}\right)\Bigg[f_1\frac{\partial^2 f_2}{\partial x^2}+f_2\left(\frac{\partial^2 f_1}{\partial x^2}\right.$$

$$\left.+2\frac{\partial^2 f_2}{\partial x\partial y}+\frac{\partial^2 f_3}{\partial y^2}\right)+f_3\frac{\partial^2 f_2}{\partial y^2}\Bigg]-2f_1f_2\left(\frac{\partial^3 f_1}{\partial x^3}+3\frac{\partial^3 f_2}{\partial x^2\partial y}\right)$$

$$-(f_2^2+2f_1f_3)\left(\frac{\partial^3 f_3}{\partial y^3}+3\frac{\partial^3 f_2}{\partial x\partial y^2}\right) \qquad \dots \quad (11\cdot 48)$$

Remembering that f_1 is a function of x only and f_3 a function of y only, it is clear that the equations (11·43) to (11·48) are all satisfied by the two equations

$$\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}=0$$

and

$$\frac{\partial f_2}{\partial x}+\frac{\partial f_3}{\partial y}=0.$$

These equations give

$$\frac{d^2 f_1}{dx^2}=-\frac{\partial^2 f_2}{\partial x\partial y}=\frac{d^2 f_3}{dy^2}.$$

Since $\frac{d^2 f_1}{dx^2}$ cannot contain y and $\frac{d^2 f_3}{dy^2}$ cannot contain x, each of these expressions must be a constant.

Hence
$$\frac{d^3 f_1}{dx^3}=0, \quad \frac{d^3 f_3}{dy^3}=0$$

$$\frac{\partial^3 f_2}{\partial x^2\partial y}=0, \quad \frac{\partial^3 f_2}{\partial x\partial y^2}=0.$$

It follows that f_1, f_2 and f_3 must be of the forms

$$f_1=ax^2+2bx+d$$

$$f_2=-2(axy+by+cx+e)$$

$$f_3=ay^2+2cy+f$$

or

$$af_1=(ax+b)^2+a'$$
$$af_2=-2(ax+b)(ay+c)+2b'$$
$$af_3=(ay+c)^2+c'$$

where $$a'=d-b^2,\ b'=bc-e,\ c'=f-c^2.$$

12. Let us now put $a'=b'=c'=0$.

Then ϕ becomes $a\{(ax+b)y'-(ay+c)\}^{-3}$.

It may be easily verified that this value of ϕ satisfies the equations (II) and (III).

Thus we get from (5·2) and (7·1),

$$F_1=ay_1^{-3}\{(ax+b)y'-(ay+c)\}^3r_1$$
$$=\frac{a}{h}y_1^{-3}(ax_1+by_1+c_1)^3r_1$$

where $$c_1=-ph.$$

This value of F_1 is obviously of the form (v) of Suchar.

13. Let us next put $a=b=c=0$, so that ϕ becomes

$$a(a'y'^2+2b'y'+c')^{-\frac{3}{2}}.$$

This expression for ϕ is also found to satisfy the equations (II) and (III).

In this case we get,

$$F_1=ay_1^{-3}(a'y'^2+2b'y'+c')^{\frac{3}{2}}r_1$$
$$=\frac{a}{h^3}y_1^{-3}(a'y_1^2+2b_1y_1+c_2)^{\frac{3}{2}}r_1,$$

where $$b_1=b'h,\quad c_2=c'h^2.$$

Here F_1 is of the form (vi) of Suchar.

14. We have thus deduced the laws (iii) to (vi) directly from the differential equations (I) to (IV) by considering some particular solutions only, and it has been pointed out that the laws (vii) and ($viii$) can be obtained from these equations by assuming ϕ to be independent of y'. It will be my endeavour to discuss the equations more fully in a subsequent paper in which I hope to be able to give the complete analytical solution of the problem.

My attention was drawn to Suchar's paper by Prof. G. Prasad.

Notes and Correspondence

Dr. S. K. Banerji, D.Sc., Ghosh Professor of Applied Mathematics, our previous Secretary, has joined the Meteorological Department of the Government of India and has been placed in charge of the Observatories in Bombay. He had to resign the Secretaryship of the Calcutta Mathematical Society, the duties of which he was discharging so ably and efficiently. Though we feel his absence keenly, we congratulate him on his appointment because he will now have ample opportunities of studying practically the problems of Earthquakes which were engaging his attention lately We offer him our sincere thanks for the past services he rendered to the Society and we hope that he will continue to take the same amount of interest in the affairs of the Society in the midst of his new sphere of activities.

The members of our Society will be glad to know that the Calcutta Mathematical Society has invited the Biennial Conference of the Mathematical Societies in India to Calcutta. The Conference will be held during the next Easter Holidays. This is the first time that we have ventured to invite the conference and we expect that the members of the Society will do their utmost to make it a complete success.

The following is an extract from the American Mathematical Monthly, Vol. XXVIII, Nov. 11-12, 1921, p. 458.

"The concluding number of the *Bulletin of the Calcutta Mathematical Society*, Vol. XI (262 pages) was published in March, 1921. This periodical, devoted to higher mathematics, pure and applied, is excellently printed and edited. At the close of 1920, the Calcutta Mathematical Society had 170 ordinary members and 25 honorary members. During 1920, 32 papers were read and the published accounts show a surplus—a state of affairs which most mathematical societies must envy."

We are much thankful to the Editor of the Journal for the good account which he has given of us. For the excellent printing of the Bulletin, we are indebted to the Calcutta University Press and we are glad to note that the attention and care they take for the printing of the Bulletin has been justly appreciated. As for the editing, credit is due to Dr. S. K. Banerji, our previous Secretary.

72, Srigopal Mallik Lane,
Bowbazar, Calcutta,
The 21st July, 1922.

To

The EDITOR,

Bulletin of the Calcutta Mathematical Society.

Sir,

In a letter dated the 26th May, 1922, Professor A. Sommerfeld informed me that the relativity correction for Zeeman-effect, which I added in the last Art. of my paper on the subject in the Bulletin No. 2, Vol. 12, of the Society, was anticipated by him in the Physikalische Zeitschrifte in 1916. Accordingly I acknowledge herein the priorty of his work, while I remark by way of explanation that my knowledge of the subject was derived from his "Atom-bau," where there was no mention of his having published the same.

I remain,
Yours faithfully,
Panchanan Das, M.Sc.

On Circular Vortex rings of finite section in incompressible fluids.

BY

Nripendranath Sen, M.Sc.

The steady motion of a circular vortex ring in an incompressible fluid has been investigated by many eminent mathematicians including Helmholtz,[1] Kelvin,[2] Lewis,[3] Thomson,[4] Basset,[5] Hicks,[6] Chree,[7] and Dyson.[8] But a complete solution of the problem has not been obtained up till now. Assuming the cross-section of the ring to be circular and very small in comparison with its aperture so that by Maxwell's electrical analogy the whole matter may be supposed to be condensed on the circular axis, Lewis, Thomson, Basset and Chree have found the velocity of a ring of constant vorticity to be

$$\frac{k}{4\pi c}\left(\log\frac{8c}{a}-1\right)$$

where k=strength of the whole vortex, c=radius of the circular axis, and a=radius of the cross-section. In a note to Prof. Tait's translation of Helmholtz's paper, Lord Kelvin gives the velocity to be

$$\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\tfrac{1}{4}\right)$$

This latter result was also given by Basset, Hicks and Dyson for a ring whose vorticity varies directly as the distance from the axis of

[1] Helmholtz—"Ueber Integrale der hydrodynamischen Gleichungen Welche den Wirbelbewegungen entsprechen." Crelle, Vol. 55, 1858.

[2] Kelvin—"Collected Scientific papers" Vol. 4, p. 67.

[3] Lewis—"Quart. Jour. Math." Vol. 16, 338-47, 1879.

[4] J. J. Thomson—"Motion of Vortex rings."

[5] Basset—Hydrodynamics, Part II.

[6] Hicks—"Phil. Trans. A Vol. 175 I, 1884. Also Vol. 176 II, 1885.

[7] Chree—"Proc. Edin. Math. Soc." Vol. 6, 1888.

[8] Dyson—"Potential of an anchor ring" Part I and II, Phil. Trans. A Vol. 184, 1893.

Also Gray—"Notes on Hydrodynamics" Phil. Mag. (6) Vol. 28, 13, 1914.

Lamb "Hydrodynamics" Ed. IV, 1916.

the ring. This discrepancy was tried to be explained by Dr. Chree who pointed out that the hypothesis of constant vorticity is not consistent with that of a truly circular section in which case he proved that the vorticity must vary inversely as the square of the distance from the axis of the ring, but he has not investigated the motion of vortex rings of finite section either of constant vorticity or of vorticity varying according to the law found out by him.

In the present paper, first the motion of vortex rings of finite section and constant vorticity has been solved and it has been shewn that the cross-section does not remain circular but is given by an equation of the form

$$r=a\left[1-\frac{36\lambda+25}{96}\frac{a^2}{c^2}\cos 2\theta-\frac{360\lambda+155}{3072}\frac{a^3}{c^3}\cos 3\theta\ldots\right]$$

where $\lambda=\log\frac{8c}{a}-2$.

As a first approximation the velocity of translation of a ring of constant vorticity and circular cross-section has been found to be

$$\frac{k}{4\pi c}\left[\log\frac{8c}{a}-\tfrac{3}{4}\right]$$

The slight difference in this result and that obtained by Thomson, Lewis and Chree has been explained at length in Art. 7 where it has been shewn that my result is perfectly accurate to the order of approximation adopted.

Secondly, the cross-section, velocity of translation and fluted oscillations of a vortex ring of finite section and vorticity varying as any power of the distance from the axis of the ring have been investigated, from which I have shewn that it is only when the vorticity varies inversely as the square root of the distance that the velocity to a first approximation is given by

$$\frac{k}{4\pi c}\left(\log\frac{8c}{a}-1\right).$$

When the vorticity obeys Chree's law referred to above the corresponding results have also been deduced from our general results; it has been shewn that even in this case the cross section does not remain circular but gets elongated in the direction of motion and the velocit to a first approximation has been found to be

$$\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\tfrac{1}{4}\right).$$

I have also shewn how the results of Hicks and Dyson are easily deducible as a particular case of the general problem here studied.

My best thanks are due to Dr. Bibhutibhusan Datta for the great interest he has taken in the preparation of this paper.

PRELIMINARY REMARKS AND DEFINITIONS.

2. Let the "circular axis" of the vortex ring be defined to be the circle passing through the centre of area of the cross sections of the vortex filament and let the perpendicular to the plane of the circular axis through its centre be called the "axis" of the vortex. Also let

$2\,\omega$=vorticity, k=strength of the vortex

c=radius of the circular axis

ρ, ϕ, z=cylindrical co-ordinates of any point referred to the centre of the circular axis as origin and the axis of the ring as z-axis.

r=distance of any point from the circular axis

θ=inclination of this distance to the plane of the circular axis so that $\rho=c-r\cos\theta$.

V=velocity of the ring parallel to z—axis

$$J=\int_0^{\pi}\frac{c\cos\phi\, d\phi}{\{z'^2+c^2-2c\rho'\cos\phi+\rho'^2\}^{\frac{1}{2}}}$$

a=mean radius of the cross-section,

$$l=\log\frac{8c}{r}-2,\ s=\frac{r}{c}$$

$$\lambda=\log\frac{8c}{a}-2,\ \sigma=\frac{a}{c}$$

$$\nabla^2=\frac{d^2}{dc^2}+\frac{d^2}{ds'^2},\ \frac{d}{dc}=\nabla\cos\alpha,\quad \frac{d}{ds'}=\nabla\sin\alpha$$

ψ=stokes's stream function,

then, it is well-known that

$$\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}=0 \text{ outside the vortex filament} \quad \dots \quad (1)$$

and

$$\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}=-2\omega\rho \text{ inside the filament} \qquad \ldots \quad (2)$$

The value of ψ at any point (ρ', ϕ', z') outside the filament may easily be seen to be given by

$$\psi=\frac{\rho'}{2\pi}\iiint\frac{\omega\,\rho\cos\phi\,d\rho\,dz\,d\phi}{\{(z'-z)^2+\rho'^2-2\rho\rho'\cos\phi+\rho^2\}^{\frac{1}{2}}} \qquad \ldots \quad (3)^1$$

where the integral is to be taken throughout the volume of the vortex filament

CASE I. ω=CONSTANT.

3. It has been pointed out by Dr. C. Chree[2] that the hypothesis of constant vorticity is not consistent with the assumption that the vortex ring has a truly circular section, but he has not discussed what the cross-section should be if the vorticity be constant. Let us take up the case here.

Let the (r, θ) equation of the cross-section of the filament be

$$r=a\,(1+A_1\cos\theta+A_2\cos 2\theta+A_3\cos 3\theta+\ldots)\ldots$$

From the definition of the circular axis, $\displaystyle\int_0^{2\pi}\!\!\int_0^{r} r^2\cos\theta\,dr\,d\theta=0.$

i.e. $\qquad A_1+A_2\,A_3+\ldots=0$

We shall presently prove that A_2 and A_3 are of the order σ^2 and σ^3 respectively.[3] A_1 is, therefore, of the order σ^5. Hence, neglecting σ^4 and higher powers of σ, the cross-section of the filament is given by

$$r=a\,(1+A_2\cos 2\theta+A_3\cos 3\theta) \qquad \ldots \quad (4)$$

Putting x for $(r\cos\theta)$ in (3), we have

$$\psi=\frac{\omega\rho'}{2\pi}\iiint\frac{(c-x)\,d\rho\,dz\,d\phi}{\{(z'-z)^2+\rho'^2-2\rho'(c-x)\cos\phi+(c-x)^2\}^{\frac{1}{2}}}$$

[1] Basset—Hydrodynamics t II, p. 60.

[2] "Vortex rings in a compressible fluid" ibid., p. 62.

[3] See results (28) and (29).

taken throughout the volume of the ring

$$=\frac{\omega\rho'}{\pi}\iint e^{-x\frac{d}{dc}-z\frac{d}{dz'}}\,dx\,dz\int_0^{\pi}\frac{c\cos\phi\,d\phi}{\{z'^2+\rho'^2-2\rho' c\cos\phi+c^2\}^{\frac{1}{2}}}$$

where the first integral is to be taken over the cross-section given by (4)

$$=\frac{\omega\rho'}{\pi}\iint e^{-x\frac{d}{dc}-z\frac{d}{dz'}}\,dx\,dz\,(\mathrm{J})\quad\ldots\quad\ldots\quad\ldots\quad(5)$$

Now, $\iint e^{-x\frac{d}{dc}-z\frac{d}{dz'}}\,dx\,dz$ taken over the cross-section (4)

$$=\int_0^{\mathrm{R}}\int_0^{2\pi} e^{-r\nabla\cos(\theta-a)}\,r\,dr\,d\theta \text{ where } \mathrm{R}=a\,(1+\mathrm{A}_2\cos 2\theta+\mathrm{A}_3\cos 3\theta)$$

$$=\int_0^{a}\int_0^{2\pi} e^{-r\nabla\cos(\theta-a)}\,r\,dr\,d\theta+\int_a^{\mathrm{R}}\int_0^{2\pi} e^{-r\nabla\cos(\theta-a)}\,r\,dr\,d\theta$$

But, $\displaystyle\int_0^{a}\int_0^{2\pi} e^{-r\nabla\cos(\theta-a)}\,r\,dr\,d\theta$

$$=\int_0^{a}\int_0^{2\pi}\{\mathrm{I}_0(r\nabla)-2\mathrm{I}_1(r\nabla)\cos(\theta-a)+2\mathrm{I}_2(r\nabla)\cos 2(\theta-a)-\text{etc.}\}\,r\,dr\,d\theta$$

where I_n is Bessel's function[1] of the nth order with imaginary modulus

$$=2\pi\int_0^{a}\mathrm{I}_0(r\nabla)\,r\,dr=2\pi\frac{\mathrm{I}_1(a\nabla)}{\nabla}\,a$$

$$=\pi a^2\left(1+\frac{a^2\nabla^2}{8}+\frac{a^4\nabla^4}{192}+\frac{a^6\nabla^6}{9216}+\frac{a^8\nabla^8}{73280}+\text{etc.}\right)$$

[1] Whittaker—"Modern Analysis" 17·7 and 17·1 Ex. 2.

Also, $\displaystyle\int_a^R\int_0^{2\pi} e^{-r\nabla_0\cos(\theta-a)}\, r\, dr\, d\theta$

$2\pi a\,(A_2\cos 2\theta+A_3\cos 3\theta)$

$$=\int_0\int_0 e^{-(r+a)\nabla\cos(\theta-a)}(a+r)\,dr\,d\theta$$

$2\pi a\,(A_2\cos 2\theta+A_3\cos 3\theta)$

$$=\int_0\int_0 e^{-a\nabla\cos(\theta-a)}\{1-r\nabla\cos(\theta-a)+\ldots\}\,(a+r)\,dr\,d\theta$$

$$=a^2\int_0^{2\pi} e^{-a\nabla\cos(\theta-a)}\,(A_2\cos 2\theta+A_3\cos 3\theta)\,d\theta$$

neglecting σ^4 and higher powers of σ

$$=a^2\int_0^{2\pi}\{I_0(a\nabla)-2I_1(a\nabla)\cos(\theta-a)+2I_2(a\nabla)\cos 2(\theta-a)-2I_3(a\nabla)\cos 3(\theta-a)+\text{etc.}\}$$

$$\times(A_2\cos 2\theta+A_3\cos 3\theta)\,d\theta$$

$$=2\pi a^2[A_2 I_2(a\nabla)\cos 2a-A_3 I_3(a\nabla)\cos 3a]$$

$$=\frac{\pi a^2}{4}\left\{A_2 a^2\nabla^2\cos 2a\left(1+\frac{a^2\nabla^2}{12}+\frac{a^4\nabla^4}{384}+\ldots\right)\right.$$

$$\left.-A_3\frac{a^3\nabla^3}{6}\left(1+\frac{a^2\nabla^2}{16}+\ldots\right)\right\}$$

$\therefore \displaystyle\iint e^{-x\frac{d}{dc}-z\frac{d}{d'z}}\, dx\, dz$ taken over the cross-section (4).

$$=\pi a^2\left\{1+\frac{a^2\nabla^2}{8}+\frac{a^4\nabla^4}{192}+\frac{a^6\nabla^6}{9216}+\text{etc.}\right.$$

$$+\frac{A_2}{4}a^2\nabla^2\cos 2\alpha\left(1+\frac{a^2\nabla^2}{12}+\frac{a^4\nabla^4}{384}+\text{etc.}\right)$$

$$-\frac{A_3}{24}a^3\nabla^3\left(1+\frac{a^2\nabla^2}{16}+\text{etc.}\right)\cos 3\alpha\Big\} \quad \ldots \quad \ldots \quad (6)^1$$

4. Sir F. W. Dyson has obtained by a different method an expression[1] for the above integral $\iint e^{-x\frac{d}{dc}-z\frac{d}{dz'}}\,dx\,dz$, which differs from that obtained by me in the co-efficient of A_3, he getting $\frac{A_3}{12}$ in the place of $\frac{A_3}{24}$ in (6). After carefully going through his calculations, I find that the difference is due to a mistake in calculation on his part, as by repeating his own process, I have got the expression (6).

5. From (5) and (6), at any point (ρ', ϕ', z') outside the filament,

$$\psi=\omega\rho' a^2\Big[1+\frac{a^2\nabla^2}{8}+\frac{a^4\nabla^4}{192}+\frac{a^6\nabla^6}{9216}+\frac{a^8\nabla^8}{73280}+\text{etc.}$$

$$+\frac{A_2 a^2}{4}\left(2\frac{d^2}{dc^2}-\nabla^2\right)\left(1+\frac{a^2\nabla^2}{12}+\frac{a^4\nabla^4}{384}+\text{etc.}\right)$$

$$-\frac{A_3 a^3}{24}\left(4\frac{d^3}{dc^3}-3\nabla^2\frac{d}{dc}\right)\left(1+\frac{a^2\nabla^2}{16}+\text{etc.}\right)\Big]J \quad \ldots \quad (7)$$

Now, it can be easily shewn that

$$\frac{d^2J}{dc^2}+\frac{d^2J}{dz'^2}-\frac{1}{c}\frac{dJ}{dc}=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)^2$$

$$\text{i.e. } \nabla^2 J=\left(\frac{1}{c}\frac{d}{dc}\right)J$$

[1] *Loc. cit.* Part II, p. 1083.

[2] This is evident from (31). Since ψ in (31) must satisfy equation (1) *viz.* $\frac{\partial^2\psi}{\partial z'^2}+\frac{\partial^2\psi}{\partial\rho'^2}-\frac{1}{\rho'}\frac{\partial\psi}{\partial\rho'}=0$ it is easy to see by writing c for ρ' and ρ' for c both in this equation and the expression for ψ *viz.* $\frac{k\rho' J}{2\pi}$ given in (31) that $\frac{d^2J}{dc^2}+\frac{d^2J}{dz'^2}-\frac{1}{c}\frac{dJ}{dc}=0$.

$$\therefore \nabla^4 J=\nabla^2\left(\frac{1}{c}\frac{d}{dc}\right)J=\frac{1}{c}\frac{d}{dc}\nabla^2 J-\frac{2}{c}\left(\frac{1}{c}\frac{d^2}{dc^2}-\frac{1}{c^2}\frac{d}{dc}\right)J$$

$$=\left(\frac{1}{c}\frac{d}{dc}\right)^2 J-2\left(\frac{1}{c}\frac{d}{dc}\right)^2 J=1\,(-1)\left(\frac{1}{c}\frac{d}{dc}\right)^2 J$$

Similarly $\nabla^6 J=1\,(-1)\,(-3)\left(\frac{1}{c}\frac{d}{dc}\right)^3 J$

... ...

$$\nabla^{2n} J=1\,(-1)\,(-3)\ldots(3-2n)\left(\frac{1}{c}\frac{d}{dc}\right)^n J \quad \ldots \quad \ldots \quad (8A)$$

where n is any positive integer.

$$\text{Also } \frac{d^2}{dc^2}=c^2\left(\frac{1}{c}\frac{d}{dc}\right)^2+\left(\frac{1}{c}\frac{d}{dc}\right) \quad \ldots \quad \ldots \quad (9)$$

$$\frac{d^3}{dc^3}=c^3\left(\frac{1}{c}\frac{d}{dc}\right)^3+3c\left(\frac{1}{c}\frac{d}{dc}\right)^2 \quad \ldots \quad \ldots \quad (9A)$$

$$\text{Again, } J\frac{\rho'}{c}=l-\frac{l+1}{2}\,s\cos\theta+\left(\frac{2l+5}{16}-\frac{l}{16}\cos 2\theta\right)s^2$$

$$+\left(\frac{3l+5}{64}\cos\theta-\frac{3l-1}{192}\cos 3\theta\right)s^3+\left(\frac{12\,l+11}{2048}\right.$$

$$\left.+\frac{12l+17}{768}\cos 2\theta-\frac{15l-8}{3072}\cos 4\theta\right)s^4+\text{etc.} \quad \ldots \quad (10)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)J=\frac{1}{c^2 s}\left\{-\cos\theta+\left(\frac{2l+3}{4}+\frac{\cos 2\theta}{4}\right)s\right.$$

$$+\left(\frac{4l+1}{32}\cos\theta+\frac{\cos 3\theta}{32}\right)s^2+\left(-\frac{4l+7}{128}+\frac{4l+1}{64}\cos 2\theta\right.$$

$$\left.\left.+\frac{\cos 4\theta}{128}\right)s^3+\text{etc.}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^2 J=\frac{1}{c^4 s^2}\left\{\cos 2\theta-\frac{\cos\theta+\cos 3\theta}{4}\,s-\left(\frac{12l+9}{32}\right.\right.$$

$$\left.\left.+\frac{\cos 2\theta}{4}+\frac{\cos 4\theta}{32}\right)s^2-\text{etc.}\right\} \quad \ldots \quad (12)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^3 J=-\frac{1}{c^6 s^3}\left\{2\cos 3\theta+\left(\cos 2\theta-\frac{\cos 4\theta}{2}\right)s+\ldots\right\} \quad (13)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^4 J=\frac{1}{c^8 s^4}\left\{6\cos 4\theta+\ldots\right\} \quad \ldots \quad \ldots \quad (14)$$

From (4), on the surface of the ring $s=\frac{r}{c}=\sigma\,(1+A_2\cos 2\theta$
$+A_3\cos 3\theta)$.

Hence, from the above relations (10) to (14), we have on the surface of the ring,

$$\frac{\rho'}{c}J=\lambda+\frac{2\lambda+5}{16}\sigma^2+\left(-\frac{\lambda+1}{2}+\frac{3\lambda+5}{64}\sigma^2-\frac{\lambda}{4}A_2\right)\sigma\cos\theta$$
$$-\left(\frac{\lambda\sigma^2}{16}+A_2\right)\cos 2\theta-\left(\frac{3\lambda-1}{192}\sigma^3+A_3+\frac{\lambda\sigma A_2}{4}\right)\cos 3\theta \quad (15)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)J=\frac{1}{c^2\sigma}\left\{\frac{2\lambda+3}{4}\sigma-\frac{4\lambda+7}{128}\sigma^3+\left(-1+\frac{4\lambda+1}{32}\sigma^2\right.\right.$$
$$\left.+\frac{A_2}{2}\right)\cos\theta+\left(\frac{\sigma}{4}+\frac{4\lambda+1}{64}\sigma^3-\frac{\sigma}{2}A_2+\frac{A_3}{2}\right)\cos 2\theta$$
$$\left.+\left(\frac{\sigma^2}{32}+\frac{A_2}{2}\right)\cos 3\theta+\left(\frac{\sigma^3}{128}+\frac{A_3}{2}\right)\cos 4\theta\right\} \quad \ldots \quad (16)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^2 J=\frac{1}{c^4\sigma^2}\left\{-\frac{12\lambda+9}{32}\sigma^2-A_2-\frac{\sigma}{4}\cos\theta\right.$$
$$\left.+\cos 2\theta\left(1-\frac{\sigma^2}{4}\right)-\frac{\sigma}{4}\cos 3\theta-\left(\frac{\sigma^2}{32}+A_2\right)\cos 4\theta\right\} \quad (17)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^3 J=\frac{1}{c^6\sigma^3}\left\{-2\cos 3\theta-\sigma\cos 2\theta+\frac{\sigma}{2}\cos 4\theta\right\} \quad (18)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^4 J=\frac{6}{c^8\sigma^4}\cos 4\theta \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

The expressions (10) to (14) have been taken from Dyson's memoir[1]
From the above relations it is easy to see that

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^n J \text{ is of the order } \frac{1}{c^{2n}\sigma^n}$$

[1] *Loc. cit.* Part I, p. 54, Part II, p. 1086-87.

Hence, $a^r c^{2n-r} \frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^n$ J is of the order σ^{r-n} ... (19A)

Also $k = \iint 2\omega r\,dr\,d\theta$ taken over the cross-section of the filament

$= 2\pi\omega a^2$ neglecting σ^4 and higher powers of σ ... (20)

Hence, always neglecting s^4 and higher powers of s, we have from (7), after a little simplification with the help of above equations, that ψ at any point (r, θ) outside the filament and very near to it,

$$= \frac{k\rho'}{2\pi}\left[1+\frac{a^4}{2}(A_2+\tfrac{1}{4})\left(\frac{1}{c}\frac{d}{dc}\right)+\left(-\frac{a^4}{192}+\frac{a^2c^2}{2}A_2\right)\left(\frac{1}{c}\frac{d}{dc}\right)^2\right.$$

$$\left.+\left(\frac{a^6}{3072}-\frac{a^4c^2}{6}A_2+\frac{a^4c^4}{24}A_2\right)\left(\frac{1}{c}\frac{d}{dc}\right)^3\right]J \quad \ldots \quad (21)$$

Whence, by (15) to (19), we have on the surface of the ring,

$$\psi = \frac{kc}{2\pi}\left[\text{constant}-\left(\frac{4\lambda+5}{8}+\frac{4\lambda+5}{16}A_2-\frac{3\lambda+4}{48}\sigma^2\right)\sigma\cos\theta\right.$$

$$-\left(\frac{A_2}{2}+\frac{12\lambda-5}{192}\sigma^2\right)\cos 2\theta-\left(\tfrac{1}{3}A_3+\frac{12\lambda+7}{48}\sigma A_2\right.$$

$$\left.\left.+\frac{8\lambda-5}{512}\sigma^3\right)\cos 3\theta\right] \quad \ldots \quad (22)$$

Since, the vortex ring moves with a velocity of translation V parallel to z-axis, we have

$\psi = \frac{V\rho^2}{2}$ + const. on the surface of the ring

$$= \frac{Vc^2}{2}\left[\text{const.}-(2+A_2)\ \sigma\cos\theta+\frac{\sigma^2}{2}\cos 2\theta-\sigma A_3\cos 3\theta\right]$$

from (4), neglecting σ^4 ond higher powers of σ ... ... (23)

Since the relation (23) is to be true for all values of θ, we have from (22) and (23) by equating the co-efficients of cos θ, cos 2θ, cos 3θ respectively.

$$Vc\left(1+\frac{A_2}{2}\right)=\frac{k}{\pi}\left\{\frac{4\lambda+5}{16}\left(1+\frac{A_2}{2}\right)-\frac{3\lambda+4}{96}\sigma^2\right\} \quad \ldots \quad (24)$$

$$Vc\,\sigma^2 = -\frac{k}{\pi}\left(A_2 + \frac{12\lambda-5}{96}\,\sigma^2\right) \quad \ldots \quad (25)$$

$$Vc\,\sigma\,A_2 = \frac{k}{\pi}\left(\tfrac{2}{3}\,A_3 + \frac{12\lambda+7}{48}\,\sigma\,A_2 + \frac{8\lambda-5}{512}\,\sigma^3\right) \quad \ldots \quad (26)$$

From (24), dividing both sides by $1+\dfrac{A_2}{2}$ and neglecting $A_2\,\sigma^2$, we have

$$V = \frac{k}{\pi c}\left(\frac{4\lambda+5}{16} - \frac{3\lambda+4}{96}\,\sigma^2\right) \quad \ldots \quad (27)$$

Substituting this value of V in (25), we have

$$A_2 = -\frac{36\lambda+25}{96}\,\sigma^2 \quad \ldots \quad (28)$$

Substituting the values of V and A_2 in (26), we obtain

$$A_3 = -\frac{360\lambda+155}{3072}\,\sigma^3 \quad \ldots \quad (29)$$

Hence, from (4), the (r, θ) equation of the cross-section is

$$r = a\left[1 - \frac{36\lambda+25}{96}\,\sigma^2\cos 2\theta - \frac{360\lambda+155}{3072}\,\sigma^3\cos 3\theta\right] \quad \ldots \quad (30)$$

From (30), it is evident that the ring does not remain circular but gets elongated in the direction of motion $\left(\theta = \dfrac{\pi}{2}\right)$ and that it may be regarded circular when and only when we neglect quantities of the order σ^2 and higher powers of σ.

6. In the previous Art. we have found out the values of V, A_2, A_3 correct to σ^3. The same method may be applied to find the velocity and shape of the cross-section correct to any power of σ, but the values of A_2, A_3 obtained in (28) and (29) give the shape of the cross-section to a fair degree of approximation even in the case of very thick rings.

Thus when $\sigma = \cdot 3$; $V = \dfrac{k}{\pi c} \times \cdot 626$; $A_2 = -\cdot 067$; $A_3 = -\cdot 005$

$\sigma = \cdot 25$; $V = \dfrac{k}{\pi c} \times \cdot 674$; $A_2 = -\cdot 051$; $A_3 = -\cdot 003$

$\sigma = \cdot 2$; $V = \dfrac{k}{\pi c} \times \cdot 733$; $A_2 = -\cdot 036$; $A_3 = -\cdot 002$

$\sigma = \cdot 1$; $V = \dfrac{k}{\pi c} \times \cdot 907$; $A_2 = -\cdot 012$; $A_3 = -\cdot 0003$

CRITICISM OF THE RESULTS OBTAINED BY SIR J. J. THOMSON AND OTHERS.

7. If the vortex ring be so thin that σ^2 and higher powers are negligible, we have found that the cross-section may be regarded as circular and in that case from (27) we have,

$$V=\frac{k}{16\pi c}(4\lambda+5)=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\tfrac{3}{4}\right).$$

But the expression for the velocity of translation of such small rings obtained by Thomson,[1] Lewis,[2] and Chree[3] is

$$V=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-1\right).$$

Thus these two velocities differ by a quantity of the order $\frac{k}{c}$. Let us investigate into the cause of this difference.

Both Sir J. J. Thomson[4] and Dr. Chree[5] have, following the common practice in calculating the velocity, regarded the vortex filament as concentrated in the circular axis. We shall presently prove that this cannot be done without committing an error of the order $\frac{k}{c}$ in the expression for the velocity.

For, if it be supposed that the cross-section of the filament is circular and very small and if it be further supposed by Maxwell's electrical analogy that "the action of a vortex ring of this shape will be the same as one of equal strength condensed at the central line of the vortex core," the corresponding expression for ψ may be easily written down from (3) by putting

$s=0$, $\rho=c$, $2\omega\, d\rho\, ds=k=$strength of the vortex.

Hence, in the present case ψ at any point (ρ', ϕ', z') is given by

$$\psi=\frac{k\rho'}{2\pi}\int_0^{\pi}\frac{c\cos\phi\, d\phi}{\{z'^2+\rho'^2-2c\rho'\cos\phi+c^2\}^{\frac{1}{2}}}$$

$$=\frac{k\rho'J}{2\pi}=\frac{kc}{2\pi}\left[l-\frac{l+1}{2}\,s\cos\theta+\text{etc.}\right]\text{ from (10)} \qquad \ldots \quad (31)^{6}$$

[1] *Loc. cit.* p. 33, Equation (41).

[2] *Loc. cit.* p. 338-47.

[3] *Loc. cit.* p. 61, Equation (14).

[4] *Loc. cit.* p. 18, Art. 8, para. I.

[5] *Loc. cit.* p. 59, para. 2.

[6] Evidently this is true whether the vorticity be constant or variable.

If the ring has a velocity of translation V parallel to z—axis. we have,

$$\psi=\frac{V\rho^2}{2}+\text{const}$$

on the surface of the ring $r=a$, *i.e.*,

$$\frac{k\rho' J}{2\pi}=\frac{V\rho^2}{2}+\text{const}$$

when $r=a$,

$\therefore$ From (31), putting $s=\sigma$, $l=\lambda$, we have,—

$$\frac{kc}{2\pi}\left\{\lambda-\frac{\lambda+1}{2}\sigma\cos\theta+\ldots\right\}=\frac{Vc^2}{2}\left[\text{constant}-2\sigma\cos\theta+\frac{\sigma^2}{2}\cos 2\theta\right]$$

Neglecting σ^2 and higher powers of σ in the above equation, we have by equating the co-efficients of $\sigma\cos\theta$ in both sides,

$$V=\frac{k}{4\pi c}(\lambda+1)=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-1\right)$$

which is the expression for the velocity obtained by Prof. J. J. Thomson, Dr. C. Chree, and others.

Now, the expression for ψ for a circular ring of very small circular cross-section may be easily deduced from our exact equation (21) by omitting A_2 and A_3, whence ψ for such a ring is given by

$$\psi=\frac{k\rho'}{2\pi}\left[1+\frac{a^2}{8}\left(\frac{1}{c}\frac{d}{dc}\right)-\frac{a^4}{192}\left(\frac{1}{c}\frac{d}{dc}\right)^2+\frac{a^6}{3072}\left(\frac{1}{c}\frac{d}{dc}\right)^3\right]J \quad (32)$$

If $\psi=\frac{k\rho' J}{2\pi}$ as given by (31) by following the method of treatment of Sir J. J. Thomson and others, evidently we reject 2nd, 3rd and other terms in the expression for ψ given by (32). But remembering that it is by equating the co-efficients of $\sigma\cos\theta$ in both sides of the equation $\psi=\frac{V\rho^2}{2}$ +const. on the surface of the ring, that the expression for the velocity is to obtained, it is absolutely necessary that all the terms in ψ given by (32), correct to first power of $\sigma\cos\theta$ must be retained. From (19 A),

$$\frac{\rho'}{c}a^2\left(\frac{1}{c}\frac{d}{dc}\right)J$$

is at the least of the order σ and must be retaind. Similarly,

$$\frac{\rho'}{c}a^4\left(\frac{1}{c}\ \frac{d}{dc}\right)^2 J$$

and

$$\frac{\rho'}{c}a^6\left(\frac{1}{c}\ \frac{d}{dc}\right)^3 J$$

are of the order σ^3 and σ^3 respectively and may be rejected. Hence the *correct* expression for ψ for a thin circular vortex ring is given by

$$\psi=\frac{k\rho'}{2\pi}\left[1+\frac{a^2}{8}\left(\frac{1}{c}\ \frac{d}{dc}\right)\right]J \quad \ldots \quad (33)$$

$$=\frac{kc}{2\pi}\left\{l-\frac{4l+5}{8}\,s\cos\theta+\text{etc.}\right\}\text{ from (10) and (11)} \quad \ldots \quad (34)$$

Since, $\psi=\frac{V\rho^2}{2}+\text{const}$ when $r=a$, we have from (34),

$$\frac{kc}{2\pi}\left\{\lambda-\frac{4\lambda+5}{8}\sigma\cos\theta+\text{etc.}\right\}=\frac{Vc^2}{2}\left[\text{const}-2\sigma\cos\theta+\ldots\right]$$

for all volues of θ.

Whence, equating the co-efficients of $\sigma\cos\theta$, we have the correct expression for the velocity given by

$$V=\frac{k}{16\pi c}\left(4\lambda+5\right)=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\frac{3}{4}\right)$$

which is the expression already obtained by us.[1]

Case II. ω variable.

8. Dr. Chree[2] has proved that in a ring of circular cross-section, the vorticity is not constant but varies according to the law

$$\omega=\Omega\rho^{-2}\quad\text{where }\Omega=\text{constant.}\quad\ldots\quad(35)^3$$

[1] Also, See. Art 12, where it has been proved that $V=\frac{k}{4\pi c}(\log\frac{8c}{a}-1)$ only when vorticity $\infty\rho^{-\frac{1}{2}}$.

[2] Loc. cit., p. 62.

[3] See Art 11.

Let us make a more general assumption *viz.*,

$$\omega = \Omega\rho^n \text{ where } n \text{ is any quantity positive or negative} \quad \dots \quad (36)$$

Hence, from (3) ψ at any point (ρ', ϕ', z') outside the filament is given in this case by

$$\psi = \frac{\rho'\Omega}{2\pi}\iiint \frac{(c-x)^{n+1}\cos\phi\, d\rho\, dz\, d\phi}{\{(z'-z)^2+\rho'^2-2(c-x)\rho'\cos\phi+(c-x)^2\}^{\frac{1}{2}}}$$

where the integral is to be taken throughout the volume of the filament

$$= \frac{\Omega\rho'}{\pi}\iint e^{-x\frac{d}{dc}-z\frac{d}{dz'}}\, dx\, dz \int_0^\pi \frac{c^{n+1}\cos\phi\, d\phi}{\{z'^2+\rho'^2-2c\rho'\cos\phi+c^2\}^{\frac{1}{2}}}$$

where the first integral is to be taken throughout any cross-section of the ring

$$= \frac{\Omega\rho'}{\pi}\iint e^{-r\nabla\cos(\theta-\alpha)}\, r\,dr\,d\theta\,(c^n \mathrm{J}) \quad \dots \quad (37)$$

where the integral is to be taken over any cross-section of the ring.

It will be proved that even in this case, the cross-section is not circular but is given by the equation (4) if σ^4 and higher powers of σ be neglected, A_2 and A_3 being quantities of the order σ^2 and σ^3 respectively.[1]

$\therefore$ From (6) and (37),

$$\begin{aligned}\psi = \Omega\rho' a^2\Big[\, 1 &+ \frac{a^2\nabla^2}{8} + \frac{a^4\nabla^4}{192} + \frac{a^6\nabla^6}{9216} + \text{etc.} \\ &+ \frac{\mathrm{A}_2}{4}\, a^2\nabla^2\cos 2\alpha\left(1+\frac{a^2\nabla^2}{12}+\frac{a^4\nabla^4}{384}+\text{etc.}\right) \\ &- \frac{\mathrm{A}_3}{24}\, a^3\nabla^3\cos 3\alpha\left(1+\frac{a^2\nabla^2}{16}+\text{etc.}\right)\Big]\, c^n\mathrm{J} \quad \dots \quad (38)\end{aligned}$$

Now, from (8)

$$\nabla^2 \mathrm{J} = \left(\frac{1}{c}\,\frac{d}{dc}\right)\mathrm{J}$$

$$\nabla^2 c^n\mathrm{J} = c^n\nabla^2\mathrm{J} + 2nc^{n-1}\frac{d\mathrm{J}}{dc} + n(n-1)c^{n-2}\mathrm{J}$$

$$= c^n\left[\,(2n+1)\left(\frac{1}{c}\,\frac{d}{dc}\right) + \frac{n(n-1)}{c^2}\,\right]\mathrm{J} \quad \dots \quad (39)$$

[1] See results (48) and (49).

Similarly,

$$\nabla^4 c^n J = c^n \left[(2n+1)(2n-1)\left(\frac{1}{c}\frac{d}{dc}\right)^2 + 2(2n-1)n(n-1)\frac{1}{c^2}\left(\frac{1}{c}\frac{d}{dc}\right) + \frac{n(n-1)(n-2)(n-3)}{c^4}\right] J \quad \ldots \quad (39A)$$

$$\nabla^6 c^n J = c^n \left[(2n+1)(2n-1)(2n-3)\left(\frac{1}{c}\frac{d}{dc}\right)^3 + \frac{3(2n-1)(2n-3)n(n-1)}{c^2}\left(\frac{1}{c}\frac{d}{dc}\right)^2 + 3(2n-3)n(n-1)(n-2)(n-3)\frac{1}{c^4}\left(\frac{1}{c}\frac{d}{dc}\right) + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{c^6}\right] J \quad \ldots \quad (39B)$$

$$\nabla^8 c^n J = c^n \left[(2n+1)(2n-1)(2n-3)(2n-5)\left(\frac{1}{c}\frac{1}{dc}\right)^4 + \frac{4(2n-1)(2n-3)(2n-5)n(n-1)}{c^2}\left(\frac{1}{c}\frac{d}{dc}\right)^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-7)}{c^8}\right] J \quad \ldots \quad (39C)$$

etc. etc.

$$a^2\nabla^2\cos 2\alpha\left(1+\frac{a^2\nabla^2}{12} + \text{etc.}\right)(Jc^n)$$

$$= a^2\left(2\frac{d^2}{dc^2} - \nabla^2\right)\left(1+\frac{a^2\nabla^2}{12} + \text{etc.}\right)(Jc^n)$$

$$= c^n\left[(2n+1)a^2\left(\frac{1}{c}\frac{d}{dc}\right) + 2a^2c^2\left(\frac{1}{c}\frac{d}{dc}\right)^2 + \frac{a^4c^2}{6}(2n+1)\left(\frac{1}{c}\frac{d}{dc}\right)^3\right] J \quad \ldots \quad (39D)$$

neglecting σ^2 and higher powers of σ.

Also $$a^3\nabla^3\cos 3a\left(1+\frac{a^2\nabla^2}{16}+\text{etc.}\right)(c^n\mathrm{J})$$

$$=a^3\left(4\frac{d^3}{dc^3}-3\frac{d}{dc}\nabla^2\right)\left(1+\frac{a^2\nabla^2}{16}+\text{etc.}\right)(\mathrm{J}c^n)$$

$$=4c^n a^3 c^3\left(\frac{1}{c}\ \frac{d}{dc}\right)^3\mathrm{J} \quad \ldots \quad (39\mathrm{E})$$

neglecting σ and higher-powers of σ.

Also $$k=\iint 2\omega r\,dr\,d\theta \text{ taken over any cross-section}$$

$$=2\Omega c^n a^2\pi\left\{1+\frac{n(n-1)}{8}\sigma^2\right\} \quad \ldots \quad (40)$$

neglecting σ^4 and higher powers of σ.

Hence, always neglecting s^4 and higher powers of s, we have from (38) after a little simplification with the help of above equations, that ψ at any point (r,θ) outside the filament and very near to it,

$$=\Omega\rho' a^2 c^n\Big[\left\{1+\frac{n(n-1)}{8}\sigma^2\right\}\mathrm{J}+\left\{\frac{2n+1}{8}a^2+\frac{a^4(2n-1)n(n-1)}{96c^2}\right.$$

$$\left.+\frac{2n+1}{4}\mathrm{A}_2 a^2\right\}\frac{1}{c}\ \frac{d\mathrm{J}}{dc}+\left\{\frac{a^4}{192}(2n+1)(2n-1)\right.$$

$$\left.+\frac{\mathrm{A}_2 a^2 c^2}{2}\right\}\left(\frac{1}{c}\ \frac{d}{dc}\right)^2\mathrm{J}+\left\{\frac{a^6}{9216}(2n+1)(2n-1)(2n-3)\right.$$

$$\left.-\frac{\mathrm{A}_3}{6}a^3c^3+\mathrm{A}_2\frac{a^4c^2}{24}(2n+1)\right\}\left(\frac{1}{c}\ \frac{d}{dc}\right)^3\mathrm{J}\Big] \quad \ldots \quad (41)$$

Hence, substituting the values of $\frac{\rho'}{c}\mathrm{J},\left(\frac{1}{c}\ \frac{d}{dc}\right)\mathrm{J}\cdot\frac{\rho'}{c}$ etc. from (15) to (19) in (41) ψ at any point on the surface of the ring

$$=\frac{kc}{2\pi}\Big[\left\{\text{const}+\left(-\frac{\lambda+1}{2}+\frac{3\lambda+5}{64}\sigma^2-\frac{\lambda}{4}\mathrm{A}_2\right)\sigma\cos\theta\right.$$

$$\left.-\left(\frac{\lambda\sigma^2}{16}+\mathrm{A}_2\right)\cos 2\theta-\left(\frac{3\lambda-1}{192}\sigma^3+\mathrm{A}_3+\frac{\lambda\sigma\mathrm{A}_2}{4}\right)\cos 3\theta\right\}$$

$$+\frac{\sigma}{8}\left\{(2n+1)-\frac{(2n+5)n(n-1)}{24}\sigma^2+2(2n+1)\mathrm{A}_2\right\}$$

$$\times\left\{\text{const}+\left(-1+\frac{4\lambda+1}{32}\sigma^2+\frac{A_2}{2}\right)\cos\theta+\frac{\sigma}{4}\cos 2\theta\right.$$

$$\left.+\left(\frac{A_2}{2}+\frac{\sigma^2}{32}\right)\cos 3\theta\right\}+\left\{\frac{\sigma^2}{192}(4n^2-1)+\frac{A_2}{2}\right\}\cdot$$

$$\times\left\{\text{const}-\frac{\sigma}{4}\cos\theta+\left(1-\frac{\sigma^2}{4}\right)\cos 2\theta-\frac{\sigma}{4}\cos 3\theta\right\}$$

$$+\left\{\frac{\sigma^3}{4608}(4n^2-1)(2n-3)-\frac{A_3}{3}+\frac{A_2\sigma(2n+1)}{12}\right\}(-\cos 3\theta)\Bigg].$$

$$=\frac{kc}{2\pi}\left[\text{const}-\left\{\frac{4\lambda+2n+5}{8}+A_2\frac{4\lambda+6n+5}{16}\right.\right.$$

$$\left.-\frac{12(n+2)\lambda+4n^3+4n^2-7n+32}{384}\sigma^2\right\}\sigma\cos\theta-\left\{\frac{A_2}{2}\right.$$

$$\left.+\frac{12\lambda-5-12n-4n^2}{192}\sigma^2\right\}\cos 2\theta-\left\{\frac{2A_3}{3}+\frac{12\lambda+2n+7}{48}A_2\sigma\right.$$

$$\left.\left.+\frac{72\lambda+8n^3+12n^2-38n-45}{4608}\sigma^3\right\}\cos 3\theta\right]. \qquad \dots \quad (42)$$

Since, the ring moves with a velocity of translation V parallel to z-axis, $\psi=\frac{V\rho^2}{2}$ +const. on the surface of the ring

$$=\frac{Vc^2}{2}\left[\text{const}-(2+A_2)\sigma\cos\theta+\frac{\sigma^2}{2}\cos 2\theta-\sigma A_2\cos 3\theta\right] \text{ from (4).} \quad (43)$$

Since, the equation (43) is to hold good for all values of θ, we have from (42) and (43) by equating the co-efficients of $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$ respectively,

$$Vc\left(1+\frac{A_2}{2}\right)=\frac{k}{2\pi}\left\{\frac{4\lambda+2n+5}{8}+A_2\frac{4\lambda+6n+5}{16}\right.$$

$$\left.-\frac{12(n+2)\lambda+4n^3+4n^2-7n+32}{384}\sigma^2\right\} \qquad \dots \quad (44)$$

$$Vc\sigma^2=-\frac{k}{\pi}\left\{A_2+\frac{12\lambda-5-12n-4n^2}{96}\sigma^2\right\} \qquad \dots \quad (45)$$

$$Vc\sigma A_2=\frac{k}{\pi}\left\{\frac{2}{3}A_3+\frac{12\lambda+2n+7}{48}A_2\sigma\right.$$

$$\left.+\frac{72\lambda+8n^3+12n^2-38n-45}{4608}\sigma^3\right\} \qquad \dots \quad (46)$$

From (45) and (46) it is evident that A_2 and A_3 are of the order σ^2 and σ^3 respectively.

Also from (44) to a first approximation,

$$Vc=\frac{k}{16\pi}\ (4\lambda+2n+5) \qquad \ldots \quad (47)$$

Substituting this value of V in (45) and simplifying

$$A_2=-\frac{36\lambda+25-4n^2}{96}\sigma^2. \qquad \ldots \quad (48)$$

Again, substituting these values of V and A_2 given by (47) and (48) respectively, in (46), we have

$$A_3=-\frac{72\lambda(2n+5)+155+62n-20-8n^3}{3072}\ \sigma^3 \qquad \ldots \quad (49)$$

Also, dividing both sides of (44) by $\left(1+\frac{A_2}{2}\right)$ and then substituting the value of A_2 as given in (48), we have,

$$V=\frac{k}{\pi c}\left\{\frac{4\lambda+2n+5}{16}-\frac{12\lambda(2n+1)+2n^2+9n+16}{384}\ \sigma^2\right\} \qquad \ldots \quad (50)$$

From (4), (48) and (49), the (r, θ) equation of the cross-section is

$$r=a\Big[\ 1-\frac{36\lambda+25-4n^2}{96}\ \sigma^2\cos 2\theta$$

$$-\frac{72\lambda(2n+5)+155+62n-20n^2-8n^3}{3072}\ \sigma^3\cos 3\theta\ \Big] \qquad \ldots \quad (51)$$

From (48), A_2 vanishes when

$$n=\pm\ \sqrt{\frac{36\lambda+25}{4}}\ .$$

Hence, it is obvious that the section of the ring may generally be regarded as circular if σ^2 and higher powers of σ are negligible. If, however,

$$n=\pm\ \sqrt{\frac{36\lambda+25}{4}}\ ,$$

the cross-section is circular correct to σ^2.

9. In Art 8, we have found out the expression for the velocity and the shape correct to σ^3 but the same process may be used to find them out correct to any higher powers of σ.

VERIFICATION OF PREVIOUS RESULTS.

10. Some particular cases are easily deducible from our general results. Thus,

(1) when $n=0$, from (50) and (51),

$$V=\frac{k}{\pi c}\left\{\frac{4\lambda+5}{16}-\frac{3\lambda+4}{96}\sigma^2\right\}$$

and $$r=a\left[1-\frac{36\lambda+25}{96}\sigma^2\cos 2\theta-\frac{360\lambda+155}{3072}\sigma^3\cos 3\theta\right].$$

These results are identical[1] with those already obtained by me.

(2) when $n=1$, from (50),

$$V=\frac{k}{\pi c}\left\{\frac{4\lambda+7}{16}-\frac{12\lambda+9}{128}\sigma^2\right\}$$

and from (51),

$$r=a\left[1-\frac{12\lambda+7}{32}\sigma^2\cos 2\theta-\frac{168\lambda+63}{1024}\sigma^3\cos 3\theta\right]$$

These results have already been obtained by Hicks[2] and Dyson.[3]

11. Now, let us deduce the results when the vorticity is given by (35) *viz.*, $\omega=\Omega\rho^{-2}$.

Putting $n=2$, from (50),

$$V=\frac{k}{\pi c}\left\{\frac{4\lambda+1}{16}+\frac{6\lambda-1}{64}\sigma^2\right\}$$

and from (51),

$$r=a\left[1-\frac{12\lambda+3}{32}\sigma^2\cos 2\theta-\frac{24\lambda+5}{1024}\sigma^3\cos 3\theta\right]$$

Hence, the cross-section of the ring gets elongated in the direction of motion also in this case.

VELOCITY OF TRANSLATION.

12. From (50), neglecting σ^2 and higher powers, we have

$$V=\frac{k}{16\pi c}(4\lambda+2n+5)$$

[1] See results (27) and (30).

[2] Loc. cit. Part I and II.

[3] Loc. cit. Part II.

Now, (1) when $n=0$, evidently

$$V=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\frac{3}{4}\right)$$

(2) when $n=1$,

$$V=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\frac{1}{4}\right)$$

(3) when $n=-2$,

$$V=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\frac{7}{4}\right)$$

(4) To find n such that

$$V=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-1\right)=\frac{k}{4\pi c}(\lambda+1),$$

we have

$$\lambda+1=\frac{4\lambda+2n+5}{4} \quad i.e.,\ n=-\frac{1}{2} \qquad \ldots \quad (52)$$

Hence, it is when vorticity $\propto\rho^{-\frac{1}{2}}$ that the velocity is given by

$$\frac{k}{4\pi c}\left(\log\frac{8c}{a}-1\right).$$

The expression for the velocity and shape of the cross-section in this case, correct to σ^3, may be easily deduced from (50) and (51).

Fluted Oscillations of the vortex ring
Vorticity $\propto\rho^n$

13. Let the central circle of the ring have moved a distance z_0 from the plane of $x\,y$ and let the cross-section in the disturbed position be given by

$$r=a\{1+\Sigma(\alpha_m\sin m\theta+\beta_m\cos m\theta)\} \qquad \ldots \quad (53)$$

Then, proceeding exactly as in Art 3 and 8, it may be obtained without much difficulty that for a ring of cross-section given by (53)

$$\psi = \frac{k\rho'}{2\pi}\left[J + \frac{(2n+1)a^2}{8}\ \frac{1}{c}\ \frac{dJ}{dc} + \text{etc.} \right.$$

$$\left. + \sum \frac{a^m}{mr^m}\ (\alpha_m \sin m\theta + \beta_m \cos m\theta) \right]$$

$$= \frac{kc}{2\pi}\left[l - \frac{l+1}{2}\ \frac{r}{c}\cos\theta - \frac{2n+1}{8}\ \frac{a^2}{cr}\cos\theta \right.$$

$$\left. + \sum \frac{a^m}{mr^m}(\alpha_m \sin m\theta + \beta_m \cos m\theta) \right] \qquad \ldots \quad (54)$$

Now,

$$\frac{dr}{dt} = \frac{\partial r}{\partial t} + \frac{\partial r}{\partial z_0}\ \frac{\partial z_0}{\partial t} + \frac{\partial r}{\partial c}\ \frac{\partial c}{\partial t}$$

$\therefore$ From (53) on the surface of the ring

$$\frac{dr}{dt} = \{1 + \sum(\alpha_m \sin m\theta + \beta_m \cos m\theta)\}\dot{a} + a\sum\{m\dot{\theta}(\alpha_m \cos m\theta$$

$$- \beta_m \sin m\theta) + (\dot{\alpha}_m \sin m\theta + \dot{\beta}_m \cos m\theta)\} \qquad \ldots \quad (55)$$

But

$$\frac{\partial \dot{r}}{\partial t} = \frac{1}{\rho r}\ \frac{\partial \psi}{\partial \theta} = \frac{k}{2\pi(c - r\cos\theta)}\ \left\{ \frac{l+1}{2}\sin\theta + \frac{2n+1}{8}\ \frac{a^2}{r^2}\sin\theta \right.$$

$$\left. + \sum \frac{a^m}{r^{m+1}}\ (\alpha_m \cos m\theta - \beta_m \sin m\theta) \right\}$$

$$= \frac{k}{2\pi c}\ \left\{ \frac{4\lambda + 2n + 5}{8}\ \sin\theta \right.$$

$$\left. + \sum \frac{c}{a}\ (\alpha_m \cos m\theta - \beta_m \sin m\theta) \right\}$$

nearly when $r=a$

$$r\dot{\theta}=-\frac{1}{\rho}\quad\frac{\partial\psi}{\partial r}=\frac{kc}{2\pi(c-r\cos\theta)}\quad\frac{1}{r}\quad\text{approximately.}$$

$$\therefore\ \dot{\theta}=\frac{k}{2\pi a^2}\quad\text{nearly when } r=a$$

Also it is easy to see that

$$\frac{\partial r}{\partial c}=\cos\theta\,;\quad\frac{\partial r}{\partial z_0}=-\sin\theta,\quad\frac{\partial\theta}{\partial c}=-\frac{\sin\theta}{r},\quad\frac{\partial\theta}{\partial z_0}=-\frac{\cos\theta}{r}$$

Hence from (55),

$$\frac{k}{2\pi c}\left\{\frac{4\lambda+2n+5}{8}\quad\sin\theta+\Sigma\frac{c}{a}\quad(a_m\cos m\theta-\beta_m\sin m\theta\right\}$$

$$-\sin\theta\frac{dz_0}{dt}+\cos\theta\dot{c}=\dot{a}+\Sigma\left\{\sin m\theta\left(\dot{a}a_m+a\dot{a}_m-\frac{km}{2\pi a}\beta_m\right)\right.$$

$$\left.+\cos m\theta\left(\dot{a}\beta_m+a\dot{\beta}_m+\frac{km}{2\pi a}a_m\right)\right\}.$$

This equation gives

$$\dot{a}=0,\quad\dot{c}=0$$

$$\dot{z}_0=\mathrm{V}=\frac{k}{4\pi c}\left(\lambda+\frac{2n+5}{4}\right)=\frac{k}{4\pi c}\left(\log\frac{8c}{a}+\frac{2n-3}{4}\right)$$

$$\dot{a}_m=\frac{k\beta_m}{2\pi a^2}(m-1)$$

$$\dot{\beta}_m=-\frac{ka_m}{2\pi a^2}(m-1)$$

i.e.,

$$\ddot{a}_m+\frac{k^2(m-1)^2}{4\pi^2a^4}\quad a_m=0$$

$$\ddot{\beta}_m+\frac{k^2(m-1)^2}{4\pi^2a^4}\quad\beta_m=0$$

$\therefore$ The oscillation is simple harmonic, the period being

$$\frac{4\pi^2 a^2}{k(m-1)}.$$

14. Conclusion.—In the preceding articles, I have studied the motion of a single vortex ring in an incompressible fluid. In a similar way the motion of any number of vortex rings can be investigated when the vorticity $\propto \rho^n$ and the results obtained by Dyson[1] may be easily deduced therefrom by putting $n=1$.

The motion of vortex-rings of finite section in a compressible fluid as well as the distribution of vorticity for which the cross-section of rings is exactly circular will be discussed in a subsequent paper.

[1] Loc. cit. ante.

2

The Steering of an Aeroplane in a Horizontal Circle.

BY

Nalinikanta Basu.

Let us start by writing down the general equations of motion of Rigid Dynamics. Taking the Centre of Mass of the aeroplane as the origin of co-ordinates and 3 rectangular axes fixed relatively to the aeroplane and moving with it in space and using the following notations :—

W, weight of the aeroplane,

A, B, C, moments of inertia about the axes,

D, E, F, corresponding products of inertia,

u, v, w, components of translational vel,

p, q, r, „ of angular vel,

h_1, h_2, h_3, „ of angular momentum,

we have the following equations of motion :—

$$\mathrm{W}\left(\frac{du}{gdt}+\frac{qw}{g}-\frac{rv}{g}\right)=\text{Acc. force along the } x\text{-axis and 2 similar}$$

equations,

also, $\dfrac{dh_1}{gdt}+\dfrac{qh_3}{g}-\dfrac{rh_2}{g}=$ Acc. torque about the x-axis and 2 similar equations and

$$h_1=\mathrm{A}p-\mathrm{F}q-\mathrm{E}r$$

$$h_2=\mathrm{B}q-\mathrm{D}r-\mathrm{F}p$$

$$h_3=\mathrm{C}r-\mathrm{E}p-\mathrm{D}q.$$

In the first place, let the aeroplane be flying steadily in a horizontal straight line. Let this be the axis of x (the line parallel to the line of flight and passing through the C G) and a line drawn vertically downwards through the C G, the y-axis and a horizontal line perpendicular to these the z-axis.

If the aeroplane be turned in any other directions the following angular co-ordinates will specify them.

Starting from an initial position, let us rotate the aeroplane about y-axis through an angle ψ and then about the new position of the z-axis through an angle θ and lastly about the final position of the x-axis through ϕ. The cosines of the angles between the old axes $x_0\ y_0\ z_0$ and the new axes $x_1\ y_1\ z_1$ are given by

	x_1	y_1	z_1
x_0	$\cos\theta\cos\psi$,	$\sin\phi\sin\psi-\cos\phi\cos\psi\sin\theta$,	$\cos\phi\sin\psi+\sin\phi\cos\psi\sin\theta$
y_0	$\sin\theta$,	$\cos\theta\cos\phi$,	$-\cos\theta\sin\phi$
z_0	$-\cos\theta\sin\psi$,	$\sin\phi\cos\psi+\cos\phi\sin\psi\sin\theta$,	$\cos\phi\cos\psi-\sin\phi\sin\psi\sin\theta$

and the angular velocities p, q, r are given in terms of $\dot\theta, \dot\phi, \dot\psi$

$$p=\dot\phi+\dot\psi\sin\theta$$

$$q=\dot\theta\sin\phi+\dot\psi\cos\theta\cos\phi$$

$$r=\dot\theta\cos\phi-\dot\psi\cos\theta\sin\phi.$$

The impressed forces and couples are due to (*i*) gravity (*ii*) the propeller thrust (*iii*) air resistances.

The components of gravity along the axes are

$$\mathrm{W}\sin\theta,\ \mathrm{W}\cos\theta\cos\phi,\ -\mathrm{W}\cos\theta\sin\phi,$$

the corresponding moments all vanishing.

The propeller thrust is assumed to act along a line parallel to the x-axis and at a point on the y-axis distant h, from the origin, then the components of thrust are

Point of application, $0, h, 0$

Force, $\mathrm{H}, 0, 0$

Torque, $0, 0, -\mathrm{H}h$.

For the components of air resistances we assume that they reduce to X, Y, Z and L, M, N and these are taken positive when they tend to retard the corresponding motions of translations and rotations.

Hence the equations of motion are in the case of a symmetrical aeroplane (in which D=E=0),

$$W\left(\frac{du}{gdt}+\frac{qw}{g}-\frac{rv}{g}\right)=W\sin\theta+H-X$$

$$W\left(\frac{dv}{gdt}+\frac{ru}{g}-\frac{pw}{g}\right)=W\cos\theta\cos\phi-Y$$

$$W\left(\frac{dw}{gdt}+\frac{pv}{g}-\frac{qu}{g}\right)=-W\cos\theta\sin\phi-Z$$

$$A\frac{dp}{gdt}-F\frac{dq}{gdt}+(C-B)\frac{rq}{g}+F\frac{pr}{g}=-L$$

$$B\frac{dq}{gdt}-F\frac{dp}{gdt}+(A-C)\frac{pr}{g}-F\frac{qr}{g}=-M$$

$$C\frac{dr}{gdt}+(B-A)\frac{pr}{g}-F\frac{p^2-q^2}{g}=-HL-N.$$

As the aeroplane is supposed to be steering steadily in a horizontal circle then initially $u=U-aQ$, $q=Q$ where Q is the angular velocity of the aeroplane in the circle and 'a' its redius and $v=w=p=r=\theta=\phi=0$, then the initial conditions of equilibrium are

$$0=H-X_0$$

$$0=W-Y_0$$

$$-\frac{W}{g}aQ^2=-Z_0$$

$$0=-L_0$$

$$0=-M_0$$

$$F\frac{Q^2}{g}=-Hh-N_0.$$

where X_0 Y_0 Z_0 L_0 M_0 N_0 are the initial values of the resistances.

Suppose then the angular velocity of the aeroplane is suddenly given an increment so that it becomes $Q+q$, the path of the aeroplane being still horizontal. So 'v' the vertical velocity is still zero. Let us also suppose that w, p, q, r, θ, ϕ be corresponding small increments, then

$U+u=(a+\rho)(Q+q)=U+aq+\rho Q$, where $a+\rho$ is the new radius

$w=\dot\rho$, $\dot\theta=r$, $\dot\phi=p$.

In the first instance we assume $F=0$, the equations are

$$W\left(\frac{du}{gdt}+\frac{Qw}{g}\right)=W\theta+H-X_0-uX_u-rX_r$$

$$Wr\frac{U}{g}=W-Y_0-uY_u-rY_r$$

$$W\left(\frac{dw}{gdt}-q\frac{U}{g}\right)=-W\phi-Z_0-wZ_w-pZ_p-qZ_q$$

$$A\frac{dp}{gdt}+(C-B)\frac{rq}{g}=-L_0-wL_w-pL_p-qL_q$$

$$B\frac{dq}{gdt}+(A-C)\frac{pr}{g}=-M_0-wM_w-pM_p-qM_q$$

$$C\frac{dr}{gdt}+(B-A)\frac{pq}{g}=-Hh-N_0-uN_u-rN_r.$$

Substituting in these $u=aq+\rho Q$, $w=\dot{\rho}$, $r=\dot{\theta}$, $p=\dot{\phi}$

$$W\left(\frac{a}{g}\frac{dq}{dt}+\frac{Q}{g}\frac{d\rho}{dt}\right)+\frac{W}{g}Q\frac{d\rho}{dt}=W\theta-(aq+\rho Q)X_u-\frac{d\theta}{dt}X_r$$

$$\frac{W}{g}aQ.\frac{d\theta}{dt}=-(aq+Q\rho)Y_u-\frac{d\theta}{dt}.Y_r$$

$$A\frac{dp}{gdt}+(C-B)\frac{Q}{g}.\frac{d\theta}{dt}=-\frac{d\rho}{dt}L_w-pL_p-qL_q$$

$$B\frac{dq}{gdt}=-\frac{d\rho}{dt}M_w-pM_p-qM_q.$$

To investigate these oscillations we assume ρ, q, θ, p each proportional to $e^{\lambda t}$ and substitute these values in the above equations and arranging we get

$$\left(\frac{W}{g}a\lambda+aX_u\right)q+\left(2\frac{W}{g}Q\lambda+QX_u\right)\rho+(\lambda X_r-W)\theta+0.p=0$$

$$aY_u.q+QY_u.\rho+\lambda\left(Y_r+\frac{W}{g}aQ\right)\theta+0.p=0$$

$$\left(\frac{B}{g}\lambda+M_q\right)q+\lambda M_w\rho+0.\theta+M_p.p=0$$

$$L_q.q+\lambda.L_w.\rho+(C-B)\frac{Q}{g}\lambda.\theta+\left(\frac{A}{g}\lambda+L_p\right)p=0.$$

Eliminating q, ρ, θ, p we get the determinant

$$\begin{vmatrix} \frac{W}{g}\,a\,\lambda+aX_u, & 2\,\frac{W}{g}\,Q\lambda+QX_u, & \lambda\,X_r-W, & 0 \\ a\,Y_u, & Q.Y_u, & (Y_r+\frac{W}{g}\,aQ)\,\lambda, & 0 \\ L_q, & \lambda L_w, & (C-B)\,\frac{Q}{g}\,\lambda, & \frac{A}{g}\,\lambda+L_p \\ \frac{B}{g}\lambda+M_q, & \lambda\,M_w, & 0, & M_p \end{vmatrix}=0.$$

Developing the determinant in powers of λ we get an equation of the 4th degree in λ

$$A\lambda^4+B\lambda^3+C\lambda^2+D\lambda+E=0$$

where
$$A=(Y_r+\frac{W}{g}\,a\,Q)\left(a\,M_w-2\,\frac{BQ}{g}\right)\frac{AW}{g^2}$$

$$B=(Y_r+\frac{W}{g}\,aQ)\,(M_w\,L_p-M_p\,L_w)\,\frac{W}{g}\,a+\frac{Aa}{g}\,M_w(X_uY_r-X_rY_u)$$

$$-\frac{AB}{g^2}\,Q\,(X_u\,Y_r-X_r\,Y_u)+\frac{W}{g}\,aQ.\frac{A}{g}\,X_u\left(a\,M_w-\frac{BQ}{g}\right)$$

$$-\left(Y_r+\frac{W}{g}\,aQ\right)(AM_q+BL_p)\,\frac{2W}{g^2}\,Q$$

$$C=-(C-B)\,\frac{Q}{g}\,M_p\,Y_u\frac{AW}{g}\,Q+(Y_r+\frac{W}{g}\,aQ)\,(M_uL_p-M_p\,L_w)\,aX_u$$

$$+\frac{2WQ}{g}\,(Y_r+\frac{W}{g}\,aQ)\,(L_q\,M_p-L_q\,M_q)-\frac{Q}{g}(AM_q+BL_p)\,(X_uY_r$$
$$-X_r\,Y_u)$$

$$+a\,Y_u\,X_r\,(L_w\,M_p-L_p\,M_w)-(AM_q+BL_p)\,\frac{W}{g}\,aQ^2\,X_u$$

$$+\frac{A}{g}\,aM_w.WY_u+\frac{ABW}{g^2}\,QY_u$$

$$D=-QX_u\,(Y_r+\frac{W}{g}\,aQ)\,(L_pM_q-L_qM_p)-WaY_u\;L_wM_p-L_pM_w)$$

$$-Q.X_rY_u\,(L_q\,M_p-L_p\,M_q)-\frac{W}{g}\,QY_u\,(AM_q+BL_p)$$

$$E=WQY_u\,(L_q\,M_p-M_q\,L_p).$$

The condition of stability require that all the four roots of the biquadratic equation for λ shall have their real part negative. This follows from the assumption that the disturbances ρ, q, θ, p are all proportional to $e^{\lambda t}$ in a typical oscillation. If λ_1 λ_2 λ_3 λ_4 are the roots the expressions for ρ, q, θ, p take the form

$$a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + a_3 e^{\lambda_3 t} + a_4 e^{\lambda_4 t}$$

a_1 a_2 a_3 a_4 being constant co-efficients determined by the initial conditions.

The conditions that the roots of a biquadratic equation shall all have their real part negative and thus indicate stability of steady motion is given by Routh. The condition is

A, B, C, D, E and F where $F = BCD - AD^2 - B^2E$, shall all have the same sign.

Now let us examine the behaviour in the above particular case of the aeroplane which Bryan has shown to possess inherent stability of great range. This is the system with 2 raised fins at the same height—two fins of areas T_1 and T_2 (of total area T) one in front and other in the rear of the C G of the system and both above the x-axis in $x-y$ plane with the y of the C P equal and their joint C P in a line through the C. G. of the system perpendicular to the main plains.

Let x, y be the co-ordinates of the Centre of mean Position (or centre of Pressure) of the 2 fins, and M_1 M_2 and P the moments and the products of inertia of the axes of the fins with respect to axes parallel to the co-ordinate axes through (x, y) we get from Bryan since $M_1 = P = 0$, in this case for the fins

$$Z_w = KTU,\ Z_p = KTUy,\ Z_q = -KTUx$$

$$L_w = KTUy,\ L_p = KTUy^2,\ L_q = -KTUxy$$

$$M_w = -KTUx,\ M_p = -KTUxy,\ M_q = KU\ (Tx^2 + M_2)$$

and by Lanchesters' Fin Resolution, $M_2 = \dfrac{T_1\ T_2}{T_1 + T_2} \cdot (\text{distance bet. fins})^2$.

Let us assume α small so that $x = 0$, and also $F = 0$, *i.e.*, the x—axis is a principle axis. Then

$$Z_w = KTU,\ Z_p = KTUy,\ Z_q = 0$$

$$L_w = KTUy,\ L_p = KTUy^2,\ L_q = 0$$

$$M_w = 0\ M_p = 0,\ M_q = kUM_2.$$

The above are the resistance derivatives due to the 2 fins only, the resistance derivatives for a main plane at an angle a and a rudder plane,

$Z_w=0,\ Z_p=0,\ Z_q=0.$

$L_w=0,\ L_p=kUI\cos^2 a,\ L_q=-2kUI\sin a\cos a$

$M_w=0,\ M_p=kUI\sin a\cos a,\ M_q=2kUI\sin^2 a.$

Hence the whole resistance derivatives are

$X_u=2ks_1\ U\sin^2 a,\ X_v=ks_1\ U\sin a\cos a,\ X_r=0$

$Y_u=2ks_1\ U\sin a\cos a,\ Y_v=ks_1\ U\cos^2 a+ks_2\ U,\ Y_r=-ks_2\ Ul$

$N_u=0,\ Nv=-ks_2\ Ul,\ N_r=ks_2Ul^2$

$Z_w=KTU,\ Z_p=kUTy,\ Z_q=0$

$L_w=kUTy,\ L_p=kUTy^2,\ L_q=-2kUI\tan a$

$M_w=0,\ M_p=-kUI\sin a\cos a,\ M_q=kUM_2.$

Substituting these values in A, B, C, D and E we get

$$\frac{A}{a^2\ Q^2}=\frac{2ABW}{ag}\left(ks_2l-\frac{W}{g}\right)\frac{AW}{g^2}$$

$$\frac{B}{a^3\ Q^3}=\left(ks_2l-\frac{W}{g}\right)k^2I\ \frac{W}{g}\ a\sin a\cos a\ Ty$$

$$+\frac{2k^2AB}{ag^2}\ s_1\sin^2 a.(s_1\cos^2 a+s_2)$$

$$-\frac{WAB}{ag^3}\ 2ks_1\sin^2 a-\frac{2W}{ag^2}\ k\left(ks_2l-\frac{W}{g}\right)(AM_2+BTy^2)$$

$$\frac{C}{a^4\ Q^4}=2\left(ks_2l-\frac{W}{g}\right)k^3s_1I\sin^3 a\cos a\ Ty\ a$$

$$-(C-B)\frac{4WI}{ag^2}\ k^2s_1\sin^2 a\cos^2 a$$

$$+\left(ks_2 l-\frac{W}{g}\right)(M_2\ Ty^2+2I^2\sin^2 a)\frac{2W}{ag}k^2$$

$$-\frac{W}{ag}\ 2k^2s_1\ (AM_2+BTy^2)\sin^2 a+\frac{2k^3}{ag}(AM_2+BTy^2)\ s_1 s_2 l\sin^2 a$$

$$-\frac{AB}{a^3gQ}\ ks_2 lW$$

$$\frac{D}{a^5Q^5}=\frac{2k^2s_1}{a}\left(ks_2 l-\frac{W}{g}\right)(QI^2\sin^2 a+M_2Ty^2)\sin^2 a$$

$$-\frac{W}{aQ^2}\,2k^3s_1\,ITy\sin^2 a\cos^2 a-\frac{W}{a^3Q^2}\ 2k^2s_1\ (AM_2+BTy^2)\sin a\cos a$$

$$\frac{E}{a^6\ Q^6}=-\frac{2Wk^3}{a^3\ Q^2}\ (2I^2\sin^2 a+M_2Ty^2)\ s_1\ \sin a\ \cos a.$$

Now from the conditions of equilibrium $W=ks_1\ U^2\sin a\cos a$ $=ks_1\ a^2\ Q^2\sin a\cos a$ and substituting these values of W in the above quantities

$$\frac{A}{U^2}=-\ \frac{2ABW}{ag^3}\quad(ks_1\ U^2\sin a\cos a/g-ks_2 l)$$

$$\frac{B}{U^3}=-\frac{KW}{g}(ks_1\ U^2\sin a\cos a/g-ks_2 l)\left\{KlaTy\sin a\cos a\right.$$

$$\left.-\frac{2(AM_2+BTy^2)}{ag}\right\}$$

$$-\frac{2kAB}{ag^2}s_1\sin^2 a\left\{\frac{W}{g}-k(s_1\cos^2 a+s_2)\right\}$$

$$\frac{C}{U^4}=-\left(\frac{ks_1U^2\sin a\cos a}{g}-ks_2 l\right)(M_2Ty^2+2I^2\sin^2 a)\ \frac{2Wk^2}{ag}$$

$$-\ \frac{AB}{a^3gQ}\ ks_2 lW-2\left(\frac{ks_1U^2\sin a\cos a}{g}-ks_2 l\right)k^3s_1 I\sin^3 a$$

$$\times\cos a.Tya-(C-B)\ \frac{4WI}{ag^2}\ k^2s_1\ \sin^2 a\cos^2 a$$

$$+\frac{2k^3}{ag}(AM_2+BTy^2)\,s_1 s_2 l \sin^2\alpha - \frac{W}{ag}\,2k^2 s_1\,(AM_2+BTy^2)\sin^2\alpha$$

$$\frac{D}{U^5} = -\frac{2k^2 s_1}{a}\,(ks_1 U^2 \sin\alpha\cos\alpha/g - ks_2 l)\,(2I^2\sin^2\alpha + M_2 Ty^2)\sin^2\alpha$$

$$-\frac{W}{aQ^2}\,2k^3 s_1\,ITy\sin^3\alpha\cos^2\alpha - \frac{W}{a^3Q^2}\,2k^2 s_1\,(AM_2+BTy^2)\sin\alpha\cos\alpha$$

$$\frac{E}{U^6} = -\frac{2k^4}{a^2}s_1\,(2I^2\sin^2\alpha + M_2 Ty_2)\sin^2\alpha\cos^2\alpha.$$

Let 'l' be positive, *i.e.*, the rudder be behind the main planes and not in front. Then A, B, C, D and E will all have the same sign as A if

$$U^2 > \frac{s_1}{s_2}\;\frac{lg}{\sin\alpha\cos\alpha}$$

and if α be a small quantity then

$$U^2 > \frac{s_1}{s_2}\,lg\cot\alpha$$

and this value of u^2 satisfies the other condition of stability $F = BCD - EB^2 - AD^2 < 0$.

Hence for stability of an aeroplane moving in a horizontal circle with its rudder plane behind its main planes,

$$U^2 > \frac{s_1}{s_2}\,lg\cot\alpha.$$

If l be negative, *i.e.*, if the rudder plane be in front of the main planes and if α be small so that we may neglect $\sin^2\alpha$ and higher powers of $\sin\alpha$

$$A/U^2 = -\frac{2ABW}{ag^2}\left(\frac{W}{g} + ks_2 l\right)$$

$$B/U^3 = -\frac{kW}{g}\left(\frac{W}{g} + ks_2 l\right)\left\{kI\alpha Ty\sin\alpha\cos\alpha - \frac{2}{ag}(AM_2+BTy^2)\right\}$$

$$C/U^4 = -\frac{2Wk^2}{ag}\left(\frac{W}{g} + ks_2 l\right) M_2 T y^2$$

$$+ \frac{AB}{a^3 g^2} k^2 s_1 s_2 l \sin \alpha \cos \alpha$$

$$D = 0$$

$$E = 0 \text{ and } F = 0.$$

Thus two roots of the biquadratic vanish. This denotes instability.

3

The Radius of a Circle in Homogeneous Co-ordinates

BY

Mohitmohan Ghosh, M.Sc.

The object of the present paper is to obtain in elegant forms the radius of a circle represented by the general equation of the second degree in homogeneous co-ordinates and the conditions for a circle as well as the co-ordinates of its centre. For the sake of simplicity the areal system of co-ordinates has been used in all the following investigations. The methods adopted are all elementary and I have purposely omitted to make use of the principle of Invariants.

First Method:

The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in areal co-ordinates is given by

$$r^2=-a^2(y_1-y_2)(z_1-z_2)-b^2(z_1-z_2)(x_1-x_2)-c^2(x_1-x_2)(y_1-y_2)$$

$\therefore$ the equation of a circle with centre at (x_1, y_1, z_1) and radius r is

$$r^2+a^2(y-y_1)(z-z_1)+b^2(z-z_1)(x-x_1)+c^2(x-x_1)(y-y_1)=0$$

or written in the homogeneous form

$$a^2yz+b^2zx+c^2xy-(x+y+z)\{a^2(yz_1+y_1z)$$
$$+b^2(zx_1+z_1x)+c^2(xy_1+x_1y)\}$$
$$+(x+y+z)^2(a^2y_1z_1+b^2z_1x_1+c^2x_1y_1+r^2)=0.$$

If this be the same as the equation

$$ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy=0$$

by comparing the co-efficients we get

$$\underset{(1)}{\frac{\mathrm{P}-(b^2z_1+c^2y_1)}{u}}=\underset{(2)}{\frac{\mathrm{P}-(c^2x_1+a^2z_1)}{v}}=\underset{(3)}{\frac{\mathrm{P}-(b^2x_1+a^2y_1)}{w}}$$

$$\underset{(4)}{\frac{2\mathrm{P}-(b^2+c^2-a^2)x_1}{2u'}}=\underset{(5)}{\frac{2\mathrm{P}-(c^2+a^2-b^2)\,y_1}{2v'}}=\underset{(6)}{\frac{2\mathrm{P}-(a^2+b^2-c^2)z_1}{2w'}}=k' \text{ (say)}$$

Where P stands for $a^2y_1z_1+b^2z_1x_1+c^2x_1y_1+r^2$.

From (2), (3) and (4) we at once get

$$k'=-\frac{a^2(x_1+y_1+z_1)}{v+w-2u'}=-\frac{a^2}{v+w-2u'}$$

Hence from symmetry we at once have as the conditions for a circle

$$\frac{v+w-2u'}{a^2}=\frac{w+u-2v'}{b^2}=\frac{u+v-2w'}{c^2}=k.$$

To find the centre we have to solve the above equations for (x_1, y_1, z_1).

From (1) (2) and (3) we have each of the above ratios equal to

$$\frac{c^2a^2\left(\frac{z_1}{c^2}+\frac{x_1}{a^2}\right)-b^2c^2\left(\frac{y_1}{b^2}+\frac{z_1}{c^2}\right)}{u-v}$$

$$=\frac{a^2b^2\left(\frac{x_1}{a^2}+\frac{y_1}{b^2}\right)-c^2a^2\left(\frac{z_1}{c^2}+\frac{x_1}{a^2}\right)}{v-w}$$

Hence we have the following equation for the centre:

$$xb^2c^2\left\{\frac{w-u}{b^2}+\frac{u-v}{c^2}\right\}+yc^2a^2\left\{\frac{u-v}{c^2}+\frac{v-w}{a^2}\right\}$$

$$+za^2b^2\left\{\frac{v-w}{a^2}+\frac{w-u}{b^2}\right\}=0 \qquad \ldots \text{(A)}$$

From (4), (5) and (6) we have similarly

$$x\frac{\cos A}{a}(v'-w')+y\frac{\cos B}{b}(w'-u')+z\frac{\cos C}{c}(u'-v')=0 \qquad \ldots \text{(B)}$$

The centre may now be found by solving (A) and (B).

Geometric meanings of (*A*) and (*B*).

It is at once seen that the above two equations will always give the centre except in the case of the circum-circle and the polar circle. Hence it may be guessed that (A) is connected in some way with the circum-circle and (B) with the polar circle. It is also seen that (A) passes through the circum-centre and (B) through the ortho-centre or the centre of the polar circle. Therefore they must be the perpendiculars upon the radical axes of the given circle with the circum-circle and the polar circle from the circum-centre and the ortho-centre respectively. This may be verified directly as follows:

The general equation may be written in the form

$$(ux+vy+wz)\,(x+y+z)-\{(v+w-2u')\,yz+(w+u-2v')zx$$
$$+(u+v-2w')\,xy\}=0$$

$$i.e.,\ \frac{1}{k}\,(ux+vy+wz)\,(x+y+z)-(a^2yz+b^2zx+c^2xy)=0$$

$\therefore$ the radical axis of the circum-circle and the given circle is

$$ux+vy+wz=0.$$

The perpendicular upon this from the circum-centre (a cosA, b cosB, c cosC) is

$$\begin{vmatrix} ua^2-vab\cos C-wca\cos B, & vb^2-wbc\cos A-uab\cos C, & wc^2-vbc\cos A-uca\cos B \\ a\cos A & b\cos B & c\cos C \\ x & y & z \end{vmatrix}=0$$

The co-efficient of x is $b\cos B\,(wc^2-vbc\cos A-uca\cos B)$

$$-c\cos C\,(vb^2-wbc\cos A-uab\cos C)$$

$$=abcu\,(\cos^2C-\cos^2B)-b^2cv\,(\cos C+\cos A\cos B)$$
$$+bc^2w\,(\cos B+\cos C\cos A)$$

$$=abcu\,(\sin^2B-\sin^2C)-b^2cv\sin A\sin B+bc^2w\sin C\sin A$$

$$=\frac{abc}{4R^2}\{u\,(b^2-c^2)-vb^2+wc^2\}\quad\text{where R=the circum-radius}$$

$$=\frac{abc}{4R^2}\,b^2c^2\left\{u\left(\frac{1}{c^2}-\frac{1}{b^2}\right)-\frac{v}{c^2}+\frac{w}{b^2}\right\}$$

$$=\frac{abc}{4R^2}\,b^2c^2\left\{\frac{w-u}{b^2}+\frac{u-v}{c^2}\right\}.$$

Hence from symmetry we see that the above equation must reduce to

$$xb^2c^2\left\{\frac{w-u}{b^2}+\frac{u-v}{c^2}\right\}+yc^2a^2\left\{\frac{u-v}{c^2}+\frac{v-w}{a^2}\right\}$$
$$+za^2b^2\left\{\frac{v-w}{a^2}+\frac{w-u}{b^2}\right\}=0$$

which is (A).

Writing the equation of the polar circle

$$bc\cos A\,x^2+ca\cos B\,y^2+ab\cos C\,z^2=0$$

in the form

$$(xbc\cos A+yca\cos B+zab\cos C)(x+y+z)-(a^2yz+b^2zx+c^2xy)=0$$

the perpendicular in this case is found to be

$$\Sigma\,(z\tan B-y\tan C)\{a^2\,(v'+w'-u')-(u'+v'-w')\,ca\cos B$$
$$-(w'+u'-v')\,ab\cos C\}=0$$

which at once reduces to (B).

The value of the radius may be found by solving the equations (1) (2)...(6) for r^2 but as the expression thus obtained is very complicated it is preferably found by the second and third methods.

SECOND METHOD :

It is evident that if we substitute the co-ordinates of a point (x_1, y_1, z_1) in the equation of a circle given in the form

$$\Sigma ux^2+2\Sigma u'yz=0.$$

We should get k times the square of the tangent from the point where k is a constant. The square of the tangent from the centre is $-r^2$ where r is the radius.

Let (x_1, y_1, z_1) be the centre and $\dfrac{x-x_1}{\lambda}=\dfrac{y-y_1}{\mu}=\dfrac{z-z_1}{\nu}=r$ the equation of any line passing through it.

Substituting in the equation of the circle we have

$$f\,(x_1, y_1, z_1)+2r\left\{\lambda\frac{\partial f_1}{\partial x_1}+\mu\frac{\partial f_1}{\partial y_1}+\nu\frac{\partial f_1}{\partial z_1}\right\}+r^2f\,(\lambda, \mu, \nu)=0.$$

As the line is drawn through the centre the co-efficient of r must be zero.

$$\therefore \lambda\frac{\partial f_1}{\partial x_1}+\mu\frac{\partial f_1}{\partial y_1}+v\frac{\partial f_1}{\partial z_1}=0 \qquad \dots \quad (1)$$

λ, μ and ν are connected by the relations

$$\lambda+\mu+v=0 \qquad \dots \quad (2)$$

$$a^2\mu\nu+b^2\nu\lambda+c^2\lambda\mu=-1 \qquad \dots \quad (3)$$

Hence we have $\dfrac{\partial f_1}{\partial x_1}=\dfrac{\partial f_1}{\partial y_1}=\dfrac{\partial f_1}{\partial z_1}$, so that $f\,(x_1, y_1, z_1)=-kr^2$.

$\therefore f(\lambda, \mu, v)=k$ where k is independent of λ, μ, v.

Thus we have the following equations:

$$u\lambda^2+v\mu^2+w\nu^2+2u'\mu\nu+2v'\nu\lambda+2w'\lambda\mu=k$$

$$\lambda+\mu+\nu=0$$

$$a^2\mu\nu+b^2\nu\lambda+c^2\lambda\mu=1.$$

There equations hold for all values of λ, μ, ν.

Putting $\lambda=0$ we get

$$v\mu^2+2u'\mu\nu+w\nu^2=k$$

$$\mu+\nu=0$$

$$a^2\mu\nu=-1.$$

Eliminating μ and ν we get the conditions for a circle in the form

$$k=\frac{v+w-2u'}{a^2}=\frac{w+u-2v'}{b^2}=\frac{u+v-2w'}{c^2}.$$

The radius is given by $-r^2=\dfrac{f(x_1, y_1, z_1)}{k}$ where (x, y, z) is the centre.

Now $f(\ _1, y_1, z_1)=\frac{1}{2}\left\{x_1\dfrac{\partial f_1}{\partial x_1}+y_1\dfrac{\partial f_1}{\partial y_1}+z_1\dfrac{\partial f_1}{\partial z_1}\right\}$

$$=-\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix} \div \begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & I & 1 & 0 \end{vmatrix}$$

$\therefore r^2=$the above quantity $\div k$.

$$\therefore r^6=\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^3 \div k^3\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^3$$

$\therefore$ from $k=\dfrac{v+w-2u'}{a^2}=\dfrac{w+u-2v'}{b^2}=\dfrac{u+v-2w'}{c^2}$ ... (1)

We have the following symmetric expression for the radius:

$$r^6=\frac{a^2b^2c^2}{(v+w-2u')\ (w+u-2v')\ (u+v-2w')}\ \frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^3}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^3} \quad \ldots \ (2)$$

$$=\frac{(4\Delta R)^2}{(v+w-2u')\ (w+u-2v')\ (u+v-2w')}\ \frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^3}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^3} \quad \ldots \ (3)$$

The equation of the nine-point-circle is

$$(b^2+c^2-a^2)\ x^2+(c^2+a^2-b^2)\ y^2+(a^2+b^2-c^2)\ z^2-2a^2yz$$
$$-2b^2zx-2c^2xy=0.$$

In this case $\Pi\ (v+w-2u')=64a^2b^2c^2$.

The discriminant$=-4a^2b^2c^2$.

Writing the co-efficients of x^2, y^2, z^2 as $2\ bc \cos A$ etc. the value of the determinant in the denominator of (3) is found to be $-64\Delta^2$.

$\therefore$ the radius of the nine point circle is given by

$$r^6=\frac{(4\Delta R)^2}{64a^2b^2c^2}\cdot\frac{4^3a^6b^6c^6}{4^9\Delta^6}=\left(\frac{R}{2}\right)^6$$

where R = the circum-radius.

$\therefore r=\frac{R}{2}$ as is well known otherwise.

THIRD METHOD:

A simple way of finding the centre, radius, etc. of the circle is to make use of the tangential equation.

Let (x_1, y_1, z_1) be the centre and r the radius of the circle

$ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy=0,$

Its tangential equation then is

$$\frac{4\Delta^2\ (lx_1+my_1+nz_1)^2}{a^2l^2+b^2m^2+c^2n^2-2mnbc\cos A-2nlca\cos B-2lmab\cos C}=r^2.$$

This is the same as $Ul^2+Vm^2+Wn^2+2U'mn+2V'nl+2W'nl=0$.

Comparing the co-efficients we get

$$\frac{4\Delta^2x^2-a^2r^2}{U}=\frac{4\Delta^2y^2+b^2r^2}{V}=\frac{4\Delta^2z^2-c^2r^2}{W}$$

$$=\frac{4\Delta^2yz+r^2bc\cos A}{U'}=\frac{4\Delta^2zx+r^2ca\cos B}{V'}=\frac{4\Delta^2xy+r^2ab\cos C}{W'}$$

$$=\frac{4\Delta^2}{U+V+W+2U'+2V'+2W'}\times\{(x^2+y^2+z^2+2yz+2zx+2xy)$$

$$-r^2(a^2+b^2+c^2-2bc\cos A-2ca\cos B-2ab\cos C)\}$$

$$=-\frac{4\Delta^2}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}}=-4\Delta^2k\ (\text{say})\quad \therefore x+y+z=1$$

and the co-efficient of $r^2=0$ identically.

Thus we get the following equations for finding the values of x, y, z, etc.:

$$x^2=\frac{a^2}{4\Delta^2}r^2-Uk \quad \dots (1) \qquad yz=-\frac{bc\cos A}{4\Delta^2}r^2-U'k \quad \dots (4)$$

$$y^2=\frac{b^2}{4\Delta^2}r^2-Vk \quad \dots (2) \qquad zx=-\frac{ca\cos B}{4\Delta^2}r^2-V'k \quad \dots (5)$$

$$z^2=\frac{c^2}{4\Delta^2}r^2-Wk \quad \dots (3) \qquad xy=-\frac{ab\cos C}{4\Delta^2}r^2-W'k \quad \dots (6)$$

Adding (1) (5) and (6) we get

$$x^2+zx+xy=\frac{r^2}{4\triangle^2}\{a^2-a\,(c\cos B+b\cos C)\}-(V'+W'+U)k$$
$$=-(V'+W'+U)k.$$

But $x^2+zx+xy=x(x+y+z)=x.$

Hence the co-ordinates of the centre are given by

$$x=-\frac{V'+W'+U}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}},\ y=-\frac{W'+U'+V}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}},\ z=-\frac{U'+V'+W}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}}$$

$$\text{or } x=\frac{V'+W'+U}{U+V+W+2U'+2V'+2W'},\ \text{etc.}$$

If we give these values to x, y and z in $f(x,\ y,\ z)$ we get the same expression for the radius as found on page (156).

Let us now try to find the conditions for a circle.

From the equations (2), (3) and (4) we have

$$16\triangle^4k^2=\frac{(4\triangle^2y^2-b^2r^2)\,(4\triangle^2z^2-c^2r^2)-(4\triangle^2yz+r^2bc\cos A)^2}{VW-U'^2}$$

$$=\frac{-4\triangle^2r^2\,(b^2z^2+2yzbc\cos A+c^2y^2)+r^4b^2c^2\sin^2 A}{VW-U'^2}.$$

Hence writing D for the discriminant of $\Sigma ux^2+2\Sigma u'yz=0$ we get from symmetry

$$16\triangle^4k^2=\frac{-4\triangle^2r^2(c^2x^2+2cazx\cos B+a^2z^2)+r^4c^2a^2\sin^2 B}{vD}$$

$$=\frac{-4\triangle^2r^2(a^2y^2+2abxy\cos C+b^2x^2)+r^4a^2b^2\sin^2 C}{wD}$$

$$=\frac{4\triangle^2r^2}{u'D}\,[\{a^2yz+zxab\cos C+xyca\cos B-x^2bc\cos A\}$$
$$+r^4a^2bc\sin B\sin C]$$

$$=\frac{-4a^2\triangle^2r^2}{(v+w-2u')D}\ \text{after some simplification.}$$

$$\text{Hence } 4\Delta^2 k^2 D = \frac{a^2 r^2}{2u'-v-w} = \frac{b^2 r^2}{2v'-w-u} = \frac{c^2 r^2}{2w'-u-v} \quad \ldots \quad (A)$$

$$= \frac{r^2 bc \cos A}{v'+w'-u-u'} = \frac{r^2 ca \cos B}{w'+u'-v-v'} = \frac{r^2 ab \cos C}{u'+v'-w-w'}$$

$$\text{Also } (v+w-2u')\,(w+u-2v')\,(v+w-2u') \cos A \cos B \cos C$$

$$=(u+u'-v'-w')\,(v+v'-w'-u')\,(w+w'-u'-v') \quad \ldots \quad (B)$$

The conditions for a circle are evident from (A).

A large number of elegent expressions for the radius may now be found from (A).

From the first three ratios of A we have

$$4\Delta^2 k^2 D = \frac{a^2+b^2+c^2}{2(u'+v'+w'-u-v-w)}\, r^2.$$

$$\therefore r^2 = \frac{8\Delta^2}{a^2+b^2+c^2}\,(u'+v'+w'-u-v-w)\,\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^2} \quad (1)$$

The same value is obtained from the last three ratios.

Multiplying the first three ratios and transposing we get from (A)

$$r^6 = -\frac{64\Delta^6 k^6 D^3 \Pi\,(v+w-2u')}{a^2 b^2 c^2} = -\frac{4\Delta^4 k^6 D^3 \Pi\,(v+w-2u')}{R^2}$$

where R = the circum-radius

$$=\left(\frac{2\Delta^2}{R}\right)(2u'-v-w)\,(2v'-w-u)\,(2w'-u-v)\,\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^3}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^6} \quad (2)$$

By (B) this result may be written as

$$r^6=\left(\frac{2\Delta^2}{R}\right)^2\frac{\{(v'+w'-u-u')\ (w'+u'-v-v')\ (u'+v'-w-w')}{\cos A\ \cos B\ \cos C}$$

$$\times\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^3}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^6} \quad \ldots \quad (3)$$

By adding the results in (A) we get

$$r^4=\tfrac{1}{3}\Delta^2\left[\frac{2u'-v-w}{a^2}+\frac{2v'-w-u}{b^2}+\frac{2w'-u-v}{c^2}\right]$$

$$\times\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^2} \quad \ldots \quad (4)$$

Writing the last three ratios in the forms

$$4\Delta^2k^2D=abc\ \frac{r^2\cos A}{a(v'+w'-u-u')}=abc\ \frac{r^2\cos B}{b(w'+u'-v-v')}$$

$$=abc\ \frac{r^2\cos C}{c(u'+v'-w-w')}$$

we get

$$r^2=\frac{4\Delta^2}{3abc}\left[\frac{a}{\cos A}(v'+w'-u-u')+\frac{b}{\cos B}(w'+u'-v-v')\right.$$

$$\left.+\frac{c}{\cos C}(u'+v'-w-w')\right]\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^2} \quad \ldots \quad (5)$$

$$=\frac{1}{3}\frac{\Delta}{R}\left[\frac{a}{\cos A}(v'+w'-u-u')+\frac{b}{\cos B}(w'+u'-v-v')\right.$$

$$\left.+\frac{c}{\cos C}(u'+v'-w-w')\right]\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}}{\begin{vmatrix} a & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}} \quad \ldots \quad (6)$$

Before finding other expressions for the radius it will be convenient to establish the following identities:

If $ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy=0$ represent a circle in areal co-ordinates

$$\{a^2u+b^2v+c^2w-2bcu'\cos A-2cav'\cos B-2abw'\cos C\}^2$$

$$=32\frac{\Delta^4}{R^2}(v+w-2u')(w+u-2v')(u+v-2w') \quad \ldots \quad (7)$$

and

$$[a^2u+b^2v+c^2w-2bcu'\cos A-2cav'\cos B-2abw'\cos C]^2$$

$$=16\Delta^2(U+V+W+2U'+2V'+2W')$$

$$=-\frac{16\Delta^2}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}} \quad \ldots \quad (8)$$

and hence

$$(v+w-2u')^2\,(w+u-2v')^2\,(u+v-2w')^2=\left(\frac{2R^2}{\Delta}\right)^2$$

$$\times\{U+V+W+2U'+2V'+2W'\}^3$$

$$=-\left[\frac{2R^2}{\Delta}\right]^2\frac{1}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^3} \quad \ldots \quad (9)$$

From the following two values for r^6 established already:

$$r^6=-\frac{16R^2\Delta^2}{\Pi(v+w-2u')}\cdot\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^3}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^3} \qquad \text{and}$$

$$r^6=-\left(\frac{2\Delta^2}{R}\right)^2\Pi(v+w-2u')\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^2}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^6}$$

we have

$$(v+w-2u')^2\ (w+u-2v')^2\ (u+v-2w')^2=-\left(\frac{2R^2}{\Delta}\right)^2 \times\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^2$$

$$=\left(\frac{2R^2}{\Delta}\right)^2[U+V+W+2U'+2V'+2W']^2.$$

which is the result (9).

Again

$$\frac{v+w-2u'}{a^2}=\frac{w+u-2v'}{b^2}=\frac{u+v-2w'}{c^2}$$

$$=\frac{u+u'-v'-w'}{bc\cos A}=\frac{v+v'-w'-u'}{ca\cos B}=\frac{w+w'-u'-v'}{ab\cos C}=\lambda$$

$$\therefore a^2u+b^2v+c^2w-2bcu'\cos A-2cav'\cos B-2abw'\cos C$$

$$=\frac{1}{\lambda}[\Sigma u\ (v+w-2u')-2\Sigma u'\ (u+u'-v'-w')]$$

$$=\frac{2}{\lambda}[U+V+W+2U'+2V'+2W'].$$

Cubing both sides and giving to λ the first three values we have

$$[\Sigma a^2u-2\Sigma bcu'\cos A]^3\,(v+w-2u')\,(w+u-2v')\,(u+v-2w')$$

$$=8a^2b^2c^2\,[U+V+W+2U'+2V'+2W']^3.$$

$\therefore$ from (9) we have

$$(a^2u+b^2v+c^2w-2bcu'\cos A-2cav'\cos B-2abw'\cos C)^3$$

$$=32\,\frac{\Delta^4}{R^2}\,(v+w-2u')\,(w+u-2v')\,(u+v-2w')$$

which is result (7).

If we square both sides of this identity and apply (9) we at once get

$$\{a^2v+b^2v+c^2w-2bcu'\cos A-2cav'\cos B-2abw'\cos C\}^2$$

$$=16\Delta^2(U+V+W+2U'+2V'+2W')=\frac{-16\Delta^2}{\begin{vmatrix} u & w' & v' & 1\\ w' & v & u' & 1\\ v' & u' & w & 1\\ 1 & 1 & 1 & 0\end{vmatrix}}$$

which is result (8).

From page (155) we have

$$r^2=\frac{a^2}{v+w-2u'}\;\frac{\begin{vmatrix} u & w' & v'\\ w' & v & u'\\ v' & u' & w\end{vmatrix}}{\begin{vmatrix} u & w' & v' & 1\\ w' & v & u' & 1\\ v' & u' & w & 1\\ 1 & 1 & 1 & 0\end{vmatrix}}$$

also $r^2 = -\dfrac{4\Delta^2}{a^2}(v+w-2u')\dfrac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^2}$

$\therefore$ multiplying we get

$$r^4 = -4\Delta^2 \frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^2}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^3} = 4\Delta^2 \frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^2}{(\mathrm{U}+\mathrm{V}+\mathrm{W}+2\mathrm{U}'+2\mathrm{V}'+2\mathrm{W}')^3} \quad \ldots \quad (10)$$

Giving to $[\mathrm{U}+\mathrm{V}+\mathrm{W}+2\mathrm{U}'+2\mathrm{V}'+2\mathrm{W}']^3$ it value from (9) we get

$$r^4 = \frac{(2\mathrm{R})^4}{[(v+w-2u')(w+u-2v')(u+v-2w')]^2}\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^2$$

or

$$r = 2\mathrm{R}\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^{\frac{1}{2}}}{[(2u'-v-w)(2v'-w-u)(2w'+u-v)]^{\frac{1}{2}}} \quad \ldots \quad (11)$$

Applying (8) we have

$$r^2 = 128\Delta^4 \frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}}{[a^2u+b^2v+c^2w-2bcu'\cos\mathrm{A}-2cav'\cos\mathrm{B}-2abw'\cos\mathrm{C}]^3} \quad \ldots \quad (12)$$

Making use of (10) we get

$$r^2 = 4R^2 \cos A \cos B \cos C \frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}}{(v'+w'-u-u')(w'+u'-v-v')(u'+v'-w-w')} \quad \dots \quad (13)$$

Several other similar expressions may be deduced by means of the identities (7), (8) and (9).

4

SURFACE WAVES DUE TO A SUBMERGED ELLIPTIC CYLINDER

BY

SUBODHCHANDRA MITRA, M.Sc.

[*Communicated by the Secretary.*]

The disturbance in the flow of a stream when a cylindrical obstacle is placed in the bed was first suggested by Kelvin.[1] But he did not calculate the actual disturbance. In a recent issue of Ann Di Matematica[2] the disturbance has been calculated when a circular cylinder is placed in the bed of a stream. The object of the present paper is to find out the disturbance in the flow of a uniform stream by a submerged elliptic cylinder placed in the bed of a stream. The method adopted in this paper can be readily extended to the case of other cylinders.

Waves due to a submerged elliptic cylinder.

Let us consider the disturbance when a cylinder whose cross-section is the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is placed athwart the stream. It is supposed that the semi-major axis and the semi-minor axis of the ellipse are small compared with f the depth of the axis of the cylinder and the origin is placed in the undisturbed level of the surface vertically above the axis.

Let us take the axis of x in the direction of flow of the stream and the axis of y vertically upwards.

We know that when liquid is streaming past a fixed elliptic cylinder, the velocity-potential[3] is given by

$$\phi=-cb\sqrt{\frac{a+b}{a-b}}e^{-\xi}\cos\eta-c\sqrt{a^2-b^2}\cosh\xi\cos\eta$$

where c is the general velocity of the stream and $x=d\cosh\xi\cos\eta$ and $y+f=d\sinh\xi\sin\eta$ and $d^2=a^2-b^2$. When there is disturbance in the

[1] Kelvin—Math and Phys. Papers t IV, p. 369 (1904).

[2] Ann. Di Matematica, (3) t XXI, p. 237 (1913).
Lamb—Hydrodynamics, 4th ed., p. 402 (1916).

[3] Lamb—Hydrodynamics, p. 80.

bed, let

$$\phi=-cb\sqrt{\frac{a+b}{a-b}}e^{-\xi}\cos\eta-c\sqrt{a^2-b^2}\cosh\xi\cos\eta+X \qquad \ldots \quad (1)$$

It can be proved easily that the normal velocity vanishes over the cylinder provided X is negligible in its neighbourhood.

Now transforming into polar coordinates (r,θ) we have from (1) after a little reduction,

$$\phi=-\frac{cax}{a-b}+\frac{cb}{2(a-b)}\left\{\left(r^2e^{2i\theta}-d^2\right)^{\frac{1}{2}}\right.$$

$$\left.+\left(r^2e^{-2i\theta}-d^2\right)^{\frac{1}{2}}\right\}+X,$$

where $x=r\cos\theta$, $y+f=r\sin\theta$.

Now making use of gamma-function,[1]

$$\int_0^\infty e^{-k(a+ib)}k^{n-1}dk=\frac{\Gamma(n)}{(a+ib)^n}.$$

Putting $n=-\frac{1}{2}$

$$\int_0^\infty e^{-k_1(a+ib)}\frac{dk_1}{k_1^{\frac{3}{2}}}=\Gamma(-\tfrac{1}{2})(a+ib)^{\frac{1}{2}}.$$

It we now write $y+f$ for a and $x+d$ for b, we have

$$\int_0^\infty e^{-k_1\{y+f+i(x+d)\}}\frac{dk_1}{k_1^{\frac{3}{2}}}=\Gamma(-\tfrac{1}{2})\{y+f+i(x+d)\}^{\frac{1}{2}}$$

$$=\Gamma(-\tfrac{1}{2})i^{\frac{1}{2}}\left(re^{-i\theta}+d\right)^{\frac{1}{2}}.$$

Also

$$\int_0^\infty e^{-k_2\{y+f+i(x-d)\}}\frac{dk_2}{k_2^{\frac{3}{2}}}=\Gamma(-\tfrac{1}{2})i^{\frac{1}{2}}\left(re^{-i\theta}-d\right)^{\frac{1}{2}}.$$

Therefore

$$\int_0^\infty\int_0^\infty e^{-(y+f)(k_1+k_2)-i\{k_1(x+d)+k_2(x-d)\}}\frac{dk_1dk_2}{k_1^{\frac{3}{2}}k_2^{\frac{3}{2}}}$$

$$=\{\Gamma(-\tfrac{1}{2})\}^2\,i\left(r^2e^{-2i\theta}-d^2\right)^{\frac{1}{2}}.$$

[1] Williamson—Integral Cal., p. 166.

Similarly

$$\int_0^\infty\int_0^\infty e^{-(y+f)(k_1+k_2)+i\{k_1(x+d)+k_2(x-d)\}}\frac{dk_1dk_2}{k_1^{\frac{3}{2}}k_2^{\frac{3}{2}}}$$

$$=\{\Gamma(-\tfrac{1}{2})\}^2\, i^3\, (r^2\, e^{2i\theta}-d^2)^{\frac{1}{2}}.$$

Thus we can write ϕ in the following way,

$$\phi=-\frac{cax}{a-b}-\frac{cb}{a-b}\int_0^\infty\int_0^\infty$$

$$\frac{e^{-(y+f)(k_1+k_2)}\sin\{(k_1+k_2)x+d(k_1-k_2)\}dk_1dk_2}{(k_1k_2)^{\frac{3}{2}}\{\Gamma(-\frac{1}{2})\}^2}+X\ldots\quad(2)$$

Let us assume,

$$X=\int_0^\infty\int_0^\infty e^{y(k_1+k_2)}\sin\{(k_1+k_2)x+d(k_1-k_2)\}\alpha(k_1,k_2)dk_1dk_2\quad\ldots\quad(3)$$

where α is a function of k_1 and k_2 to be determined. For the equation of the free surface assumed to be steady, let us put

$$\eta_1=\int_0^\infty\int_0^\infty\beta(k_1,k_2)\cos\{(k_1+k_2)x+d(k_1-k_2)\}dk_1dk_2\quad\ldots\quad(4)$$

The conditions to be satisfied at the free surface are

$$\left[\frac{\partial\phi}{\partial y}\right]_{y=0}=\frac{cd\eta_1}{dx}.\quad\ldots\quad(5)$$

Since the variable part of the pressure at the free surface will be constant,[1] *i.e.*

$$\frac{p}{\rho}=-gy-\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2,$$

or the expression

$$\frac{p}{\rho}=-g\eta_1-\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2\quad\ldots\quad(6)$$

will be independent of x provided terms of the second order are neglected.

[1] Lamb—Hydrodynamics, ibid.

From (5) we get

$$c\beta=\frac{cb}{a-b}\cdot\frac{e^{-f(k_1+k_2)}}{(k_1k_2)^{\frac{3}{2}}\{\Gamma(-\frac{1}{2})\}^2}+a \qquad \dots (7)$$

From (6) we have

$$g\beta=-\frac{c^2ab}{(a-b)^2}\frac{(k_1+k_2)e^{-f(k_1+k_2)}}{(k_1k_2)^{\frac{3}{2}}\{\Gamma(-\frac{1}{2})\}^2}+(k_1+k_2)a\frac{ca}{a-b} \qquad \dots (8)$$

From (7) and (8) we obtain

$$\beta=\frac{2b}{a-b}\frac{e^{-f(k_1+k_2)}(k_1+k_2)}{(k_1k_2)^{\frac{3}{2}}\{\Gamma(-\frac{1}{2})\}^2\left[(k_1+k_2)-\frac{g(a-b)}{c^2a}\right]} \qquad \dots (9)$$

and eliminating β

$$a=\frac{cb}{a-b}\frac{e^{-f(k_1+k_2)}\left\{(k_1+k_2)+\frac{g(a-b)}{c^2a}\right\}}{(k_1k_2)^{\frac{3}{2}}\{\Gamma(-\frac{1}{2})\}^2} \qquad \dots (10)$$

If we write k for $\dfrac{g(a-b)}{c^2a}$ we have

$$\eta_1=\frac{2b}{a-b}\int_0^\infty\int_0^\infty \frac{e^{-f(k_1+k_2)}(k_1+k_2)\cos\{(k_1+k_2)x+d(k_1-k_2)\}dk_1dk_2}{(k_1k_2)^{\frac{3}{2}}\{\Gamma(-\frac{1}{2})\}^2[k_1+k_2-k]} \qquad \dots (11)$$

η_1 may also be written in the form

$$\eta_1=\frac{2b}{a-b}\Bigg[\int_0^\infty\int_0^\infty \frac{e^{-f(k_1+k_2)}\cos\{(k_1+k_2)x+d(k_1-k_2)\}dk_1dk_2}{(k_1k_2)^{\frac{3}{2}}}$$

$$+\int_0^\infty\int_0^\infty \frac{e^{-f(k_1+k_2)}k\cos\{(k_1+k_2)x+d(k_1-k_2)\}dk_1dk_2}{(k_1k_2)^{\frac{3}{2}}[(k_1+k_2)-k]}\Bigg]$$

$$\times\frac{1}{\{\Gamma(-\frac{1}{2})\}^2} \qquad \dots (12)$$

It now remains to evaluate the integrals and determine the form of the free surface. Let us evaluate the integrals separately.

The first integral can be written in the form

$$I_1=\int_0^\infty \frac{e^{-fk_2}\cos k_2(x-d)dk_2}{k_2^{\frac{3}{2}}}\int_0^\infty \frac{e^{-fk_1}\cos k_1(x+d)}{k_1^{\frac{3}{2}}}$$

$$-\int_0^\infty \frac{e^{-fk_2}\sin k_2(x-d)dk_2}{k_2^{\frac{3}{2}}}\int_0^\infty \frac{e^{-fk_1}\sin k_1(x+d)dk_1}{k_1^{\frac{3}{2}}}$$

$$=\frac{\Gamma(-\frac{1}{2})}{\sqrt{2}}\left\{f+\{f^2+(x+d)^2\}^{\frac{1}{2}}\right\}^{\frac{1}{2}}\times\frac{\Gamma(-\frac{1}{2})}{\sqrt{2}}\left\{f+\{f^2+(x-d)^2\}^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$-\frac{\left(\Gamma(-\frac{1}{2})\right)^2}{2}\left\{\{f^2+(x+d)^2\}^{\frac{1}{2}}-f\right\}^{\frac{1}{2}}\left\{\{f^2+(x-d)^2\}^{\frac{1}{2}}-f\right\}^{\frac{1}{2}}$$

$$=\frac{\{\Gamma(-\frac{1}{2})\}^2}{2}\left[\left\{f+\left((f^2+(x+d)^2\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\left\{f+\{f^2+(x-d)^2\}^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right.$$

$$\left.-\left\{\left(f^2+(x+d)^2\right)^{\frac{1}{2}}-f\right\}^{\frac{1}{2}}\left\{\left(f^2+(x-d)^2\right)^{\frac{1}{2}}-f\right\}^{\frac{1}{2}}\right].$$

The second integral is indeterminate. But it can be reduced to an integral under a sign of integration by means of the complex variable. Let us write it in the form

$$I_2=k\int_0^\infty \frac{e^{-fk_2}\cos k_2(x-d)dk_2}{k_2^{\frac{3}{2}}}\int_0^\infty \frac{e^{-fk_1}\cos k_1(x+d)dk_1}{k_1^{\frac{3}{2}}\,(k_1+k_2-k)}$$

$$-k\int_0^\infty \frac{e^{-fk_2}\sin k_2(x-d)dk_2}{k_2^{\frac{3}{2}}}\times\int_0^\infty \frac{e^{-fk_1}\sin k_1(x+d)dk_1}{k_1^{\frac{3}{2}}\,(k_1+k_2-k)}.$$

Since the integral is an even function of x, we may consider the case only when x is positive.

The properties of the integral are contained within those of the complex integral

$$\iint \frac{e^{-fy'+iy'(x-d)}dy'\cdot e^{-fy+iy(x+d)}\,dy}{y'^{\frac{3}{2}}\;y^{\frac{3}{2}}\;(y+y'-k)}\quad \text{where } y=u+iv,\ y'=u'+iv'.$$

Now the integral has a singularity when $y_1+y'-k=0$ where y_1 is the corresponding value of y.

The complex integral

$$=\int \frac{e^{-fy'+iy'(x-d)}dy'}{y'^{\frac{3}{2}}} \cdot \int \frac{e^{-fy+iy\,(x+d)}dy}{y^{\frac{3}{2}}\,(y-y_1)}.$$

Now it is a well-known theorem[1] in the Theory of Functions that

$$\int \frac{F(y)}{f(y)} = \frac{\pi_i F(a)}{f'(a)} \text{ where } a \text{ is a simple root of } f(y)=0$$

and the integration is taken along a circle described about a so as to exclude it.

Hence the second integral

$$=\int \frac{e^{-fy'+iy'(x-d)}dy'}{y'^{\frac{3}{2}}}\,\pi i\, \frac{e^{-fy_1+iy'(x+d)}}{y'^{\frac{3}{2}}}$$

$$=\pi i \int \frac{e^{\{-fy'+iy'(x-d)-f(k-y')+i(k-y')(x+d)\}}\,dy'}{y'^{\frac{3}{2}}\,(k-y')^{\frac{3}{2}}}.$$

Now the second integral is the real part of this complex integral.

Thus the second integral becomes

$$\mathrm{I}=k\pi\int_0^\infty \frac{e^{-fk}\sin\{2k_2 d-k(x+d)\}dk_2}{k_2^{\frac{3}{2}}\,\{k-k_2)^{\frac{3}{2}}}.$$

Therefore

$$\eta_1=\frac{b}{a-b}\Bigg[\ \left\{\left(f+\{f^2+(x+d)^2\}^{\frac{1}{2}}\right)^{\frac{1}{2}}\left(f+\{f^2+(x-d)^2\}^{\frac{1}{2}}\right)^{\frac{1}{2}}\right\}$$

$$-\left\{\left(f^2+(x+d)^2\right)^{\frac{1}{2}}-f\right\}^{\frac{1}{2}}\left\{\left(f^2+(x-d)^2\right)^{\frac{1}{2}}-f\right\}^{\frac{1}{2}}$$

$$+\frac{k\pi}{\{\Gamma(-\frac{1}{2})\}^2}\int_0^\infty \frac{e^{-fk}\sin\{2k_2 d-k(x+d)\}dk_2}{k_2^{\frac{3}{2}}\,(k-k_2)} + \text{etc.}\ \Bigg] \qquad (13)$$

[1] Lamb—Hydrodynamics, p. 391, p. 398.
Also Forsyth Theory of Functions. Art 24.

Waves due to a submerged circular cylinder.

From this we can deduce the form of the free surface when a circular cylinder is placed in the bed of the stream.

Let us consider the equation (13) previously obtained

$$\eta_1=\frac{b}{a-b}\Big[\int_0^\infty\int_0^\infty e^{-f(k_1+k_2)+i\{(k_1+k_2)x+d(k_1-k_2)\}}\cdot\frac{dk_1dk_2}{(k_1k_2)^{\frac{3}{2}}}$$

$$+\int_0^\infty\int_0^\infty e^{-f(k_1+k_2)-i\{(k_1+k_2)x+d(k_1-k_2)\}}$$

$$\frac{dk_1dk_2}{(k_1k_2)^{\frac{3}{2}}}\Big]\frac{1}{\{\Gamma(-\frac{1}{2})\}^2}+\text{etc.}$$

we have seen that

$$\int_0^\infty e^{-fk_1+ik_1(x+d)}\frac{dk_1}{k_1^{\frac{3}{2}}}=\Gamma(-\tfrac{1}{2})\Big(f-i(x+d)\Big)^{\frac{1}{2}}.$$

Hence

$$\eta_1=\frac{b}{a-b}\Big[\Big(f-i(x+d)\Big)^{\frac{1}{2}}\Big(f-i(x-d)\Big)^{\frac{1}{2}}$$

$$+\Big(f+i(x+d)\Big)^{\frac{1}{2}}\Big(f+i(x-d)\Big)^{\frac{1}{2}}\Big]$$

$$=\frac{b}{a-b}\Big[(f-ix)^{\frac{1}{2}}\Big\{1-\frac{id}{f-ix}\Big\}^{\frac{1}{2}}(f-ix)^{\frac{1}{2}}\Big\{1+\frac{id}{f-ix}\Big\}^{\frac{1}{2}}$$

$$+(f+i\ )^{\frac{1}{2}}\Big\{1+\frac{id}{f+ix}\Big\}^{\frac{1}{2}}\times f+i\ )^{\frac{1}{2}}\Big\{1-\frac{id}{f+ix}\Big\}^{\frac{1}{2}}\Big]+\text{etc.}$$

Now we can neglect higher powers of d above the second. To illustrate this we see that

$$\frac{d^3}{a-b}=d(a+b)=\sqrt{a^2-b^2}(a+b)=0,\ \text{when } a=b.$$

The same thing holds for all the higher powers. So we can retain only the second powers of d. Therefore

$$\eta_1=\frac{b}{a-b}\left[(f-ix)\left(1+\frac{d^2}{2(f-ix)^2}\right)+(f+ix)\left(1+\frac{d^2}{2(f+ix)^2}\right)\right]+\text{etc.}$$

$$=\frac{b}{a-b}\left[2f+\tfrac{1}{2}d^2\left(\frac{1}{f-ix}+\frac{1}{f+ix}\right)\right]+\text{etc.}$$

$$=\frac{b}{a-b}\left[2f+\frac{fd^2}{f^2+x^2}\right]+\text{etc.}$$

neglecting an infinite constant

$$\eta_1=\frac{bf(a+b)}{f^2+x^2}+\text{etc.}$$

when $a=b$

$$\eta_1=\frac{2a^2f}{f^2+x^2}+\text{etc.}$$

This agrees with the form of the free surface for a circular cylinder.[1]

[1] Lamb—Hydrodynamics, p. 402, Art. 247.

5

AN ALGEBRAICAL IDENTITY

BY

PANDIT OUDH UPADHYAYA.

The identity $4X = Y^2 - (-1)^{\frac{p-1}{2}} pz^2$, given by Gauss for the transformation of X where X represents $\dfrac{x^p - 1}{x - 1}$ is well known. There is another identity given by Eisenstein[1] for the transformation of X in the form $27X = f(U, V, W)$.

Very recently the author of this paper has shown that any number of identities can be easily obtained, and a general method of finding out those identities has been given. There it has been shown that Gauss' identity and Eisenstein's identity are only particular cases of that general theorem. The object of this paper is to find out another formula of the eleventh degree; and it is believed that this formula has not been given by any previous writer.

Let $\eta_0, \eta_1, \eta_2, \ldots, \eta_{10}$ be the roots of the cyclotomic eleven-sectional periods and let X_1, be a polynomial of which the co-efficients are symmetric functions of the roots of $X=0$, the sum of which makes $\eta_0 = 0$. Similarly $X_2, X_3, \ldots, X_{11}$, are defined. [X_1, X_2, etc., have the same significance as has been given to them in Mathews' Theory of numbers.] Only that case has been considered in which X_1 may be represented in the form $U + V\eta_0$. Therefore for the case in consideration we have identically

$$
\begin{aligned}
X_1 &= U + V\eta_0, \\
X_2 &= U + V\eta_1, \\
&\ldots \quad \ldots \\
\text{and } X_{11} &= U + V\eta_{10},
\end{aligned}
$$

where U and V are polynomials with integral co-efficients.

[1] Mathematische Abhandlungen (Berlin 1847); No. 1, Darstelung des Ausdrucks $27 \dfrac{x^{p-1}}{x-1}$ durch eine aibische Form mit drei Variablen.

Now it is well known that

$$X = X_1 X_2 X_3 \ldots X_{11}$$

$$= (U+V\eta_0)(U+V\eta_1)(U+V\eta_2)\ldots(U+\eta V_{10})$$

$$= U^{11} + \Sigma\eta_0 U^{10}V + \Sigma\eta_0\eta_1 U^9V^2 + \Sigma\eta_0\eta_1\eta_2 U^8V^3 + \Sigma\eta_0\eta_1\eta_2\eta_3 U^7V^4$$

$$+\Sigma\eta_0\eta_1\ldots\eta_4 U^6V^5 + \Sigma\eta_0\eta_1\ldots\eta_5 U^5V^6 + \Sigma\eta_0\eta_1\ldots\eta_6 U^4V^7$$

$$+\Sigma\eta_0\eta_1\ldots\eta_7 U^3V^8 + \Sigma\eta_0\eta_1\ldots\eta_8 U^2V^9 + \Sigma\eta_0\eta_1\ldots\eta_9 UV^{10}$$

$$+\eta_0\eta_1\ldots\eta_{10}V^{11}.$$

Substituting the values of these symmetric functions we find that

$$X = U^{11} - U^{10}V + aU^9V^2 - bU^8V^3 + cU^7V^4 - dU^6V^5 + eU^5V^6$$

$$-fU^4V^7 + gU^3V^8 - hU^2V^9 + iUV^{10} - jV^{11} \qquad \ldots \quad (A)$$

where a, b, c, etc., are the co-efficients of $\eta^9, \eta^8, \ldots, \eta$ and the constant term respectively in the eleven-sectional period equation.

Let us apply this theorem to the prime 23, and thus verify the formula in this case. It is known that $U = x^2+1$ and $V = -x$ for 23; and the period equation of the cyclotomic eleven-section for the prime 23 is

$$\eta^{11} + \eta^{10} - 10\eta^9 - 9\eta^8 + 36\eta^7 + 28\eta^6 - 56\eta^5 - 35\eta^4 + 35\eta^3$$

$$+15\eta^2 - 6\eta - 1 = 0.$$

$$\therefore a = -10,\ b = -9,\ c = 36,\ d = 28,\ e = -56,\ f = -35 g = 35,\ h = 15,$$

$$i = -6 \text{ and } j = -1,$$

substituting these values in (A) we get

$$X = U^{11} - U^{10}V - 10U^9V^2 + 9U^8V^3 + 36U^7V^4 - 28U^6V^5 - 56U^5V^6$$

$$+35U^4V^7 + 35U^3V^8 - 15U^2V^9 - 6UV^{10} + V^{11}.$$

If we put $1 = x$ in the formula it becomes

$$23 = 2048 + 1024 - 5120 - 2304 + 4608 + 1792 - 1792 - 560$$

$$+280 + 60 - 12 - 1\,;$$

and thus we can represent 23 in the form of the eleventh degree with the help of this formula.

I should like to mention that I have received help in calculation from Pandit Shukdeo Chaubey.

Algebra of Polynomials

BY

Nripendranath Ghosh.

Chapter I.

Simple fundamental theorems.

1. Let $u_a(z)$ or simply u_a represent the rational and integral polynomial

$$a_0+a_1z+a_2z^2+a_3z^3+\ldots+a_nz^n$$

of the nth degree in z, whose coefficients $a_0, a_1, a_2 \ldots a_n$ are non-zero finite variables, mutually independent of one another. Let the first derivative of u_a (with regard to z) be represented by u'_a, the second by u''_a and generally the rth derivative $(r \ngtr n)$ by $u_a^{(r)}$.

2. Corresponding to the polynomial u_a, let Δ_{a0} stand for the linear differential operator

$$a_1\frac{\partial}{\partial a_0}+2a_2\frac{\partial}{\partial a_1}+3a_3\frac{\partial}{\partial a_2}+\ldots+na_n\frac{\partial}{\partial a_{n-1}}$$

where $ra_r\dfrac{\partial}{\partial a_{r-1}}$ will be called the rth term of the operator and ra_r the rth coefficient. Δ_{a0} is evidently a particular type of a more general linear differential operator

$$a_1\frac{\partial}{\partial a_p}+2a_2\frac{\partial}{\partial a_{p+1}}+3a_3\frac{\partial}{\partial a_{p+2}}+\ldots+(n-p+1)a_{n-p+1}\frac{\partial}{\partial a_n}$$

where p may have any of the values $0,1,2,3\ldots n$. We shall denote this latter operator by Δ_{ap}.

3. The operators Δ_{ap} are called simple in contradistinction to another class of linear differential operators called complex. An operator will be called complex when all its coefficients involve the variable z. In the case when some of the coefficients involve the variable z and others do not, the operator will be called mixed.

4. If $\phi(W)$ be a continuous function of W then

$$\frac{d}{dz}\phi(u_a)=\Delta_{a0}\phi(u_a).$$

In proof of this theorem we observe

$$\frac{d}{dz}u_a=u'_a=(a_1+2a_2z+3a_3z^2+\ldots+na_nz^{n-1})$$

also

$$\Delta_{a0}u_a=\left(a_1\frac{\partial}{\partial a_0}+2a_2\frac{\partial}{\partial a_1}+3a_3\frac{\partial}{\partial a_2}+\ldots+na_n\frac{\partial}{\partial a_{n-1}}\right)u_a$$

$$=a_1\frac{\partial u_a}{\partial a_0}+2a_2\frac{\partial u_a}{\partial a_1}+3a_3\frac{\partial u_a}{\partial a_2}+\ldots+na_n\frac{\partial u_a}{\partial a_{n-1}}$$

$$=a_1+2a_2z+3a_3z^2+\ldots+na_nz^{n-1}$$

so that
$$\frac{d}{dz}u_a=\Delta_{a0}u_a$$

$$\therefore \frac{d}{dz}\phi(u_a)=\frac{\partial\phi}{\partial u_a}u'_a=\frac{\partial\phi}{\partial u_a}\Delta_{a0}u_a=\Delta_{a0}\phi(u_a)$$

which proves the theorem.

5. The proof of the above theorem holds true as none of the coefficients a_0, a_1, $a_2\ldots a_n$ of u_a is zero, *i.e.*, as the polynomial is complete. Incomplete polynomials may be best treated by means of complex operators. In what follows, unless contrary is stated, the polynomials are always complete.

6. If $\phi(W)$ be any continuous function of W then

$$z^2\frac{d}{dz}\phi(u_a)=\left(\Delta_{a2}+na_nz\frac{\partial}{\partial a_n}\right)\phi(u_a)\ ;$$

for

$$z^2\frac{d}{dz}\phi(u_a)=\frac{\partial\phi}{\partial u_a}z^2u'_a=\frac{\partial\phi}{\partial u_a}\{a_1z^2+2a_2z^3+\ldots+(n-1)a_{n-1}z^n\}$$
$$+\frac{\partial\phi}{\partial u_a}na_nz^{n+1}$$

$$=\Delta_{a2}\phi(u_a)+na_nz\frac{\partial}{\partial a_n}\phi(u_a)$$

$$=\left(\Delta_{a2}+na_nz\frac{\partial}{\partial a_n}\right)\phi(u_a)$$

7. If $u_a, u_b, u_c \ldots$ represent a number of polynomials

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n,$$

$$b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m,$$

$$c_0 + c_1 z + c_2 z^2 + \ldots + c_l z^l,$$

of n, m, lth degrees respectively and if $\Delta_{a0}, \Delta_{b0}, \Delta_{c0}, \ldots$ be operators corresponding to $u_a, u_b, u_c \ldots$ respectively, then

$$\frac{d}{dz}\phi(u_a, u_b, u_c \ldots) = (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \ldots)\phi(u_a, u_b, u_c \ldots)$$

where ϕ is a continuous function of $u_a, u_b, u_c \ldots$

In proof of this theorem we observe

$$\frac{d}{dz}u_a = u'_a = \Delta_{a0} u_a = (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \cdots) u_a$$

$$\frac{d}{dz}u_b = u'_b = \Delta_{b0} u_b = (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \cdots) u_b$$

$$\frac{d}{dz}u_c = u'_c = \Delta_{c0} u_c = (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \cdots) u_c$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\therefore \frac{d}{dz}\phi(u_a, u_b, u_c \ldots) = \frac{\partial \phi}{\partial u_a} u'_a + \frac{\partial \phi}{\partial u_b} u'_b + \frac{\partial \phi}{\partial u_c} u'_c \ldots$$

$$= (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \ldots)\phi(u_a, u_b, u_c, \ldots)$$

which proves the theorem.

The operator $(\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \cdots)$ will be called a compound operator.

8. If $u_a, u_b, u_c \ldots$ represent a number of polynomials as in art 7 and if $\Delta_{a2}, \Delta_{b2}, \Delta_{c2} \ldots$ be operators corresponding to $u_a, u_b, u_c \ldots$ respectively then

$$z^2 \frac{d}{dz}\phi(u_a, u_b, u_c \ldots) = (\Delta_{a2} + \Delta_{b2} + \Delta_{c2} + \ldots + n a_n z \frac{\partial}{\partial a_n}$$

$$+ m b_m z \frac{\partial}{\partial b_m} + l c_l z \frac{\partial}{\partial c_l} \ldots)\phi(u_a, u_b, u_c \ldots)$$

where ϕ is a continuous function of $u_a, u_b, u_c \ldots$

We have from Art 6

$$z^2\frac{d}{dz}u_a=\left(\Delta_{a2}+na_nz\frac{\partial}{\partial a_n}\right)u_a$$

$$z^2\frac{d}{dz}u_b=\left(\Delta_{b2}+mb_mz\frac{\partial}{\partial b_m}\right)u_b$$

$$z^2\frac{d}{dz}u_c=\left(\Delta_{c2}+lc_lz\frac{\partial}{\partial c_l}\right)u_c\ldots$$

so that

$$z^2\frac{d}{dz}\phi(u_a,\, u_b,\, u_c,\ldots)$$

$$=\frac{\partial\phi}{\partial u_a}z^2\frac{d}{dz}u_a+\frac{\partial\phi}{\partial u_b}z^2\frac{d}{dz}u_b+\frac{\partial\phi}{\partial u_c}z^2\frac{d}{dz}u_c\ldots$$

$$=\frac{\partial\phi}{\partial u_a}\left(\Delta_{a2}+na_nz\frac{\partial}{\partial a_n}\right)u_a+\frac{\partial\phi}{\partial u_b}\left(\Delta_{b2}+mb_mz\frac{\partial}{\partial b_m}\right)u_b+$$

$$\frac{\partial\phi}{\partial u_c}\left(\Delta_{c2}+lc_lz\frac{\partial}{\partial c_l}\right)u_c\ldots$$

$$=\left(\Delta_{a2}+\Delta_{b2}+\Delta_{c2}+\ldots+na_nz\frac{\partial}{\partial a_n}+mb_mz\frac{\partial}{\partial b_m}\right.$$

$$\left.+lc_lz\frac{\partial}{\partial c_l}\ldots\right)\phi(u_a,u_b,u\ \ldots)$$

$\because$ the mixed operators are mutually independent of one another.

The operator

$$\left(\Delta_{a2}+\Delta_{b2}+\Delta_{c2}+na_nz\frac{\partial}{\partial a_n}+mb_mz\frac{\partial}{\partial b_m}+lc_lz\frac{\partial}{\partial c_l}\right)$$

is an instance of a compound mixed operator.

9. Functions of the derivatives of u_a.

We have

$$\frac{d}{dz}u_a=\Delta_{a0}u_a \quad \ldots \quad \text{(art 4)}$$

whence $\dfrac{d}{dz}u'_a=\dfrac{d}{dz}\Delta_{a0}u_a=\Delta_{a0}\dfrac{d}{dz}u_a=\Delta_{a0}u'_a$

$\therefore \dfrac{d}{dz}u''_a=\Delta_{a0}u''_a$ and so on.

If now $\phi(u_a, u'_a, u''_a \ldots u_a^{(r)})$, $r \not> n$

be a continuous function of u_a and its derivatives only then

$$\frac{d}{dz}\phi(u_a, u'_a \ldots u_a^{(r)}) = \Delta_{a0}\phi(u_a, u'_a \ldots u_a^{(r)});$$

for $$\frac{d}{dz}\phi(u_a, u'_a, u''_a \ldots u_a^{(r)})$$

$$= \frac{\partial\phi}{\partial u_a}\frac{du_a}{dz} + \frac{\partial\phi}{\partial u'_a}\frac{du'_a}{dz} + \frac{\partial\phi}{\partial u''_a}\frac{du''_a}{dz} + \ldots + \frac{\partial\phi}{\partial u_a^{(r)}}\frac{du_a^{(r)}}{dz}$$

$$= \frac{\partial\phi}{\partial u_a}\Delta_{a0}u_a + \frac{\partial\phi}{\partial u'_a}\Delta_{a0}u'_a + \frac{\partial\phi}{\partial u''_a}\Delta_{a0}u''_a + \ldots + \frac{\partial\phi}{\partial u_a^{(r)}}\Delta_{a0}u_a^{(r)}$$

$$= \Delta_{a0}\phi(u_a, u'_a, u''_a \ldots u_a^{(r)})$$

which proves the theorem.

10. If $\phi(u_a, u'_a, u''_a \ldots u_b, u'_b, u''_b \ldots u_c, u'_c, u''_c \ldots)$ be a continuous function of $u_a, u_b, u_c \ldots$ and their derivatives only, then

$$\frac{d}{dz}\phi(u_a, u'_a, u''_a \ldots u_b, u'_b, u''_b \ldots u_c, u'_c, u''_c \ldots)$$

$$= (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \ldots)\phi(u_a, u'_a, u''_a \ldots u_b, u'_b, u''_b \ldots u_c, u'_c, u''_c \ldots)$$

The proof of this theorem presents no new peculiarity. This includes theorem in Art 7 as a particular case.

11. Transformed theorems :—

Let $u_a(z)$ represent the rational and integral polynomial

$$a_0 + a_1 z + a_2 z^2 + \ldots a_n z^n$$

of the nth degree in z, then it may be subjected to two distinct types of transformation giving rise to transformed polynomials.

In the first type we change the variable z to some other variable t, connected by the explicit relation $z = \psi(t)$, so that the transformed polynomial becomes $a_0 + a_1\psi(t) + a_2(\psi t)^2 + a_3(\psi t)^3 + \ldots a_n(\psi t)^n$, represented by $u_a(\psi t)$.

As a result of the first type of transformation of polynomials all our previous theorems will necessarily lead to transformed theorems of the first type.

Let us take as an example, theorem in Art 4 viz.,

$$\frac{d}{dz}\phi(u_a)=\Delta_{a0}\phi'(u_a).$$

Put $z=\psi(t)$, then $dz=\psi'(t)dt$, so that the required transformed theorem becomes

$$\frac{1}{\psi'(t)}\cdot\frac{d}{dt}\phi\{u_a(\psi t)\}=\Delta_{a0}\phi\{u_a(\psi t)\}$$

In the second type of transformation of u_a we change only its co-efficients $a_0a_1a_2...a_n$ partly or all together. Since these coefficients are non-zero finite variables mutually independent of one another, the transformation to which u_a is subjected must be such as to preserve this characteristic.

Evidently a most general form of transformation is a combination of the two types indicated here.

2

On the Spectra of Isotopes

BY

Panchanon Das, M.Sc.

The work of Aston and Dempster has established the fact that most elements possess two or more isotopes, which have the same atomic number and arrangement of electrons, but have nuclei of slightly different masses. The question at once arises, whether the spectra of these isotopes should be different to any extent at all. Bohr's treatment of the dynamics of the hydrogen atom with a nucleus of finite mass is easily extended to any atom and leads to the result that the Rydberg constant R, instead of being an absolute constant, would vary very slightly from element to element owing to the finite mass of the nucleus. Thus isotopes with identical electrical structure, but slightly different nuclear mass, would have their respective Rydberg-constants slightly different from each other and a separation should exist between the corresponding series-lines.

This was first put to the experimental test by Merton[1] who examined the line $\lambda = ...4058$AU of ordinary lead and compared its position with that of the corresponding line of its isotope, *viz.*, the lead derived from radioactive sources. He found the actual separation to be 0·011 AU being about 100 to 200 times as large as the theoretical value of Bohr. Quite recently MacLennan[2] in course of a study of the fine structure of the Lithium line $\lambda = 6708$AU, succeeded in resolving it into four components. As theoretically this should be doublet, he attributed this circumstance to the presence of isotopes. He computed that the actual isotopic separation was about 3 to 4 times as large as the value calculated from Bohr's theory. He finally generalised that the actual separation of the spectra of any two isotopes must be atomic number times as large as the theoretical result of Bohr. So it appears that there is something fallacious in the existing theory. Ehrenfest[3] points out that the fallacy

[1] Proc. Roy. Soc., Vol. 100.
[2] Proc. Roy. Soc., Vol. 101.
[3] Nature, June 10, 1922

lies in applying the results of a two-body problem to an n-body problem. Bohr's original theory referred to the atoms of hydrogen and ionised helium only. The spectroscopic consequences of his theory were borne out remarkably well by facts. But when we come to an atom of a higher complexity, it becomes an n-body problem and Bohr's results cannot hold good One must investigate the joint influence of all the electrons on the motion of the nucleus in order to explain the existing discrepancy. An interesting side-light on this point was thrown by Silberstein[1] in a letter on the series-spectra of neutral helium. He finds that each of the electrons surrounding an atomic nucleus describes an orbit practically uninfluenced by the rest of electrons, and in deriving a series-formula, he makes use of the total-energy of the whole atom instead of the valency-electron only, as is usually done. The valency-electron, regarded as an isolated system, should apparently behave as non-holonomous, hence its total energy is generally a function of time. The best course is to quantise the generalised coordinates of the whole system of electrons and nucleus constituting the atom, as this last represents a conservative system. If we make certain simple assumptions, the prohibitive nature of the n-body problem gives way and a separation of variables in the Hamilton-Jacobi equation of motion can be effected.

Let M be the mass of the nucleus of an atom of any element, of which the atomic number is N and let g be the centre of mass of these N electrons. If the mass of an electron be m, the centre of mass G of the whole system consisting of electrons and nucleus divides the line Mg in the ratio of M ; Nm. Let $Gg=R_1$, and $GM=R_2$, and let the line Mg, which, we assume, lies in an invariable plane, make an angle θ with any fixed line in this plane.

Since the atom is not subject to any external forces, the point G may be regarded as fixed. We proceed to compute the kinetic energy T of the system. Since the centre of mass G is at rest the K·E of the system is equal to the K·E relative to G. Or T=K·E of M relative to G+K·E of the N electrons relative to G. Again the K·E of the N electrons relative to G=the K·E of the mass of N electrons relative to g. If we take time-average over a long period, this K·E of electrons relative to their own centre of mass may be regarded as zero. Thus,

$$T=\frac{1}{2}\,M(\dot{R}_2^2+R_2^2\dot{\theta}^2)+\frac{1}{2}\,Nm(\dot{R}_1^2+R_1^2\dot{\theta}^2),$$

where $R_1:R_2=M:Nm$.

[1] Nature, August 19, 1922.

Let us now put $R = R_1 + R_2$. Then,

$$R_1 = \frac{MR}{M+Nm}, \text{ and } R_2 = \frac{NmR}{M+Nm}.$$

Substituting these values in T, we get,

$$T = \frac{1}{2} \cdot \frac{Nm}{1+\frac{Nm}{M}} \cdot (\dot{R}^2 + R^2\dot{\theta}^2),$$

$$= \frac{1}{2} \mu(\dot{R}^2 + R^2\dot{\theta}^2), \text{ say}$$

where

$$\mu = \frac{Nm}{1+\frac{Nm}{M}} \quad \ldots \quad (1)$$

To find the mutual potential energy of the system we assume that all the electrons except the outermost one are grouped close together around the nucleus, so that the centre of mass g_1 of these $N-1$ electrons is very closely situated to the nucleus O, compared to the outermost electron m.

Now since $g_1 g : gm = 1 : N-1$, it is evident that Og is approximately $\frac{1}{N}$th part of Om, if we regard Og_1 as small compared with Om. But Og was previously denoted by R. Thus, $Om = NR$, approximately.

We may regard the mutual potential energy of the $N-1$ electrons near the nucleus and the nucleus itself as remaining unaffected during a radiation, so that we may omit it from our Hamilton-Jacobian equation of motion.

Next, if we assume after Silberstein that these electrons are independent of each other, the mutual potential energy of the outer electron and the rest may be disregarded also. The only effective term in the mutual potential energy V is then the potential energy of the outer electron and the nucleus. Or,

$$V = -\frac{Ee}{Om}. \quad \text{But } E = Ne \text{ and } Om = NR.$$

Thus, $V = -\frac{e^2}{R}$ approximately.

Evidently the Hamilton-Jacobian equation of motion is

$$\frac{1}{2\mu}\left\{\left(\frac{\partial S}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial S}{\partial \theta}\right)^2\right\} - \frac{e^2}{R} = W.$$

This form has been treated at length by Sommerfeld, and the total-energy W is easily seen to be

$$W = -\frac{2\pi^2 \mu e^4}{h^2} \cdot \frac{1}{n^2}$$

where n is the sum of radial and angular quantum-numbers. Now the theoretical value of Rydberg's constant R_∞ for an element with an infinite nuclear mass is

$$R_\infty = -\frac{2\pi^2 m e^2}{h^3}.$$

Hence after restoring the value of μ from (1) we get

$$W = -\frac{N \cdot R_\infty}{1 + \frac{Nm}{M}} \cdot \frac{h}{n^2}.$$

When a Quantum-transit takes place, the difference of total energy is a multiple of $h\nu$. Professor C. V. Raman suggested that since N electrons take part, this multiple might be taken to be N. Thus,

$$N \cdot h\nu = \frac{N \cdot R_\infty h}{\left(1 + \frac{Nm}{M}\right)} \left(\frac{1}{n^2} - \frac{1}{p^2}\right),$$

where p is the value of n in a different configuration.

Or, $$\nu = \frac{R_\infty}{1 + \frac{Nm}{M}} \left(\frac{1}{n^2} - \frac{1}{p^2}\right) \qquad \ldots \quad (2)$$

This is a series formula of the Balmer type. We assume that this modified Rydberg's constant will replace the usual value in the more general series-formulae also.

Now suppose that an element has got two isotopes of atomic weight M_1 and M_2 and that the frequencies of the corresponding series-lines for which the quantum-numbers are the same, are ν_1 and ν_2. Then it is easily seen after some approximations, that

$$\nu_1-\nu_2=\Delta\nu=Nm\left(\frac{1}{M_1}-\frac{1}{M_2}\right)R_\infty\left(\frac{1}{n^2}-\frac{1}{p^2}\right).$$

It is evident that this separation is N times as large as the value obtained from Bohr's formula :—

$$\nu=\frac{R_\infty}{1+\frac{m}{M}}\left(\frac{1}{n^2}-\frac{1}{p^2}\right).$$

This explains the discrepancy between observed results and values calculated from previous theories.

There are some side-issues, which call for a separate consideration. It is obvious that our modified Rydberg's constant, varies slightly from one element to another. It is not a case of steady increase or decrease with the atomic number but is of the nature of an oscillating function, as is apparent when we note that the nuclear mass M measured in terms of the mass of a hydrogen atom also varies with the atomic number and is in fact, proportional, as a first approximation, to the same. Thus N/M is a quantity of which the value oscillates between 1 and 2 for a large number of elements. If we take N/M equal to 1 for these elements then the variable Rydberg's constant becomes identical with the Rydberg constant for hydrogen. This tallies well with the remark in Fowler's Report on Series-spectra, that the series-spectra of most elements can be closely represented by means of the Rydberg constant for hydrogen.

The modified Rydberg constant was put to more rigorous test by the consideration of a particular element, helium. As the wave-lengths of the sharp series of all elements can generally be measured with the greatest accuracy, the sharp singlet series in the spectrum of helium

was chosen and the modified Rydberg constant was introduced into the Hicks formula, in order to see if any improvement could be effected. The original formula is quoted here from Fowler's Report :—

$$\nu = 1P - m\mathrm{S}$$

$$= 27175\cdot17 - \frac{109722}{\left(m + 0\cdot862157 - \frac{0\cdot010908}{m}\right)^2}$$

where $m = 2, 3, \ldots$

We first evaluate the modified Rydberg constant for helium. Its value is R_{He}, given by :—

$$R_{He} = \frac{R_\infty}{1 + \frac{Nm}{M_{He}}} = \frac{R_\infty}{1 + \frac{2\cdot m}{4M_H}}$$

$$= R_\infty\left(1 - \frac{m}{2M_H}\right) \text{ approximately,}$$

where M_{He} and M_H are the masses of a helium and a hydrogen nucleus respectively.

From Sommerfeld's Atom-bau, the value of R_∞ is taken to be 109737, and R_H is 109677. Hence, to calculate the value of

$R_\infty \times \frac{m}{M_H}$, we have,

$$R_\infty - R_H = 109737 - 109677 = 60$$

or $$R_\infty - R_\infty\left(1 - \frac{m}{M_H}\right) = R_\infty \times \frac{m}{M_H} = 60.$$

$$\therefore R_{He} = 109737 - 30 = 109707.$$

We now replace 109722 by 109707 in the mS term of Hicks formula and introduce two other constants s, σ by way of compensation as follows:—

$$\nu = 27175 \cdot 17 - \frac{109707}{\left(m + 0 \cdot 862157 + s - \frac{0 \cdot 010908 + \sigma}{m}\right)^2}$$

Putting $m=2$ and 3, we equate the mS term to 13445·23 and 7369·82 respectively and solve for S and σ from the two resulting equations.

The values are:—

$$s = -0 \cdot 000476$$

and $$\sigma = -0 \cdot 000486.$$

Thus the revised farmula is

$$\nu = 27175 \cdot 17 - \frac{109707}{\left(m + 0 \cdot 861681 - \frac{0 \cdot 010422}{m}\right)^2}.$$

The series-lines arising from this formula are tabulated below:—

m	mS observed.	mS revised.	O−C($\Delta\nu$) Hicks.	O−C($\Delta\nu$) revised.
2	13445·23	13445·23	0·00	0·00
3	7369·82	7369·82	0·00	0·00
4	4646·52	4646·51	0·00	−0·01
5	3195·17	3195·21	+0·10	+0·04
6	2331·21	2331·27	+0·14	+0·06
7	1775·25	1775·59	+0·52	+0·34
8	1397·15	1397·43	+0·35	+0·21
9	1127·91	1128·33	+0·49	+0·42
10	not observed	...	...	...
11	779·93	779·85	−0·02	−0·08
12	661·48	663·28	+1·85	+1·80

The second column gives the value of mS taken from Fowler's Report, and the third the value calculated from Hicks' formula. The symbol O—C stands for the difference $\Delta\nu$ between observed and calculated values of wave-numbers.

It will be clear from a study of the last two columns that a systematic improvement is effected by our modification. It has not been possible to try it on other elements as the process is extremely laborious, but the fact that for most elements the Rydberg constant corresponds to the value for hydrogen lends strong support to our hypothesis.

On Magnetic Field due to a Thermionic Valve.

BY

K. Basu.

1. Langmuir[1] has shown that the electrons emitted by a heated metallic filament and conveyed by means of an external electric field to a concentric cylinder (anode) produce, on account of their charges an electrostatic field which tends to limit a further discharge of electrons from the heated filament. A theory of this effect has been given by Langmuir himself, which has been later on improved by von Laue.[2] They have shown that if V is the electrostatic potential in the space occupied by electrons

$$\nabla^2 V = -4\pi\rho$$

where $\rho = Ne$ is the density of electrons at any point.

Since the electrons move radially outwards from the filament to the cylinder with a definite mass velocity it is clear that in addition to an electrostatic field we shall also have an electromagnetic field. Let (a_1, a_2, a_3) be the vector potentials defining this field, then we have

$$\nabla^2 (a_1, a_2, a_3) = -4\pi\rho\ (u_1, u_2, u_3).$$

Now if the quantities (ρ, u_1, u_2, u_3) be known and the boundary conditions be given, (a_1, a_2, a_3) can be calculated at any point. From the values of (a_1, a_2, a_3) so determined, the magnetic field can be calculated by using the relation

$$F, G, H = \text{rot } a.$$

The above is on the supposition that the phenomenon is perfectly steady. If the ejection of electrons be subject to fluctuations the time factor must be taken into account, we have then

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) V = -4\pi\rho,$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) a = -4\pi\rho u,$$

[1] Phy. Rev., Vol. II, p. 453 (1913).

Laue—Jahrbuch der drahtlose Telegraphie, 1919.

where ρ, u are both functions of time. In this case the electric and magnetic fields are given by

$$F,\ G,\ H = \text{rot } \mathbf{a},$$

$$X,\ Y,\ Z = -\text{Grad. } V - \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3).$$

2. In the present problem we need only calculate the vector-potentials given by

$$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \int (i_x, i_y, i_z)\,\frac{d\Omega}{R},$$

where i_x, i_y, i_z are components of current at any space-element $d\Omega$, R its distance from a *particular* point at which the *potentials* are taken. Then using cylindrical co-ordinates we find

$$\mathbf{a}_1 = \iiint \frac{i\cos\theta}{R}\,r dr\,d\theta\,dz = \iiint \left(\frac{a}{r}i_0\right)\frac{r\cos\theta\,dr\,d\theta\,dz}{R},$$

$$\mathbf{a}_2 = \iiint \frac{i\sin\theta}{R}\,r dr\,d\theta\,dz = \iiint \left(\frac{a}{r}i_0\right)\frac{r\sin\theta\,dr\,d\theta\,dz}{R},$$

$$\mathbf{a}_3 = 0,$$

where the axis of the filament (radius a) is taken as the z-axis and the central plane through the mid-point of the axis perpendicular to the generators of the cylinder (or filament) is taken as the plane $z=0$, $\theta=$ azimuth of the element $r\,dr\,d\theta\,dz$. If the co-ordinates of the particular point be (r_0, θ_0, z_0), we have

$$R^2 = r_0^2 + r^2 + (z_0 - z)^2 - 2rr_0\cos(\theta - \theta_0).$$

If $(\mathbf{a}_{r_0}, \mathbf{a}_{\theta_0}, \mathbf{a}_{z_0})$ be the potentials with reference to cylindrical co-ordinates and $(M_{r_0}, M_{\theta_0}, M_{z_0})$ the new magnetic intensity

$$M = \text{rot } \mathbf{a},$$

$$\begin{vmatrix} \mathbf{a}_{r_0} \\ \mathbf{a}_{\theta_0} \\ \mathbf{a}_{z_0} \end{vmatrix} = \begin{vmatrix} \cos\theta_0, & \sin\theta_0, & 0 \\ -\sin\theta_0, & \cos\theta_0, & 0 \\ 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{vmatrix}$$

Thus

$$\mathbf{a}_{r_0} = i_0 a \int_{r=a}^{b} \int_{z=-l}^{l} \int_{I=0}^{2\pi} \frac{\cos I \, dr \, dz \, d\,I}{\{r_0^2 + r^2 + (z_0 - z)^2 - 2rr_0 \cos I\}^{\frac{1}{2}}},$$

$$\mathbf{a}_{\theta_0} = i_0 a \int_{r=a}^{b} \int_{z=-l}^{l} \int_{I=0}^{2\pi} \frac{\sin I \, dr \, dz \, d\,I}{\{r_0^2 + r^2 + (z_0 - z)^2 - 2rr_0 \cos I\}^{\frac{1}{2}}},$$

putting $I = \theta - \theta_0$, in the above, the limits of integration of θ and I are the same.

The values of the integrals

$$\int_0^{2\pi} \frac{\cos I \, d\,I}{\{r_0^2 + r^2 + (z_0 - z)^2 - 2rr_0 \cos I\}^{\frac{1}{2}}},$$

$$\int_0^{2\pi} \frac{\sin I \, d\,I}{\{r_0^2 + r^2 + (z_0 - z)^2 - 2rr_0 \cos I\}^{\frac{1}{2}}}$$

are respectively $\frac{2}{(r_0 r)^{\frac{1}{2}}} \left[\left(\frac{2}{k} - k \right) F - \frac{2}{k} E \right]$ and zero, where $k^2 = 4rr_0 / (r_0 + r)^2 + (z_0 - z)^2$, and F, E, the complete elliptic integrals of the first and second kind.

Now

$$M_{r_0} = \frac{1}{r_0} \frac{\partial}{\partial \theta_0} \mathbf{a}_{z_0} - \frac{\partial}{\partial z_0} \mathbf{a}_{\theta_0},$$

$$M_{\theta_0} = \frac{\partial}{\partial z_0} \mathbf{a}_{r_0} - \frac{\partial}{\partial r_0} \mathbf{a}_{z_0},$$

$$M_{z_0} = \frac{1}{r_0} \frac{\partial}{\partial r_0} (r_0 \, \mathbf{a}_{\theta_0}) - \frac{1}{r_0} \frac{\partial}{\partial \theta_0} \mathbf{a}_{r_0}.$$

In the present case,

$$\left[M_{r_0}, M_{\theta_0}, M_{z_0} \right] = \left[0, \frac{\partial}{\partial z_0} \mathbf{a}_{r_0}, 0 \right]$$

Hence

$$M_{\theta_0}=i_0 a\,\frac{\partial}{\partial z_0}\int\limits_{r=a,}^{b}\int\limits_{z=-l}^{l}\frac{2}{(rr_0)^{\frac{1}{2}}}\left[\left(\frac{2}{k}-k\right)F-\frac{2}{k}E\right]dr\,dz.$$

3. Now

$$F=\int\limits_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}=\frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2k^4\right.$$
$$\left.+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2k^6+\dots\right],$$

$$E=\int\limits_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\psi}\,d\psi=\frac{\pi}{2}\left[1-\left(\frac{1}{2}\right)^2k^2-\left(\frac{1\cdot3}{2\cdot4}\right)^2\frac{k^4}{3}\right.$$
$$\left.-\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\frac{k^6}{5}-\dots\right].$$

As a first approximation, take $F=E=\frac{\pi}{2}$.

$$\therefore M_{\theta_0}=\tfrac{1}{2}\,\pi i_0 a\int\limits_a^b\int\limits_{-l}^{l}\frac{\partial}{\partial z_0}\left[\frac{2}{(rr_0)^{\frac{1}{2}}}\left\{\left(\frac{2}{k}-k\right)-\frac{2}{k}\right\}dr\,dz\right.$$

$$=\tfrac{1}{2}\,\pi i_0 a\int\limits_a^b\int\limits_{-l}^{l}\frac{2}{(rr_0)^{\frac{1}{2}}}\left(-\frac{\partial k}{\partial z_0}\right)dr\,dz.$$

But $-\dfrac{\partial k}{\partial z_0}=(z_0-z)k/(r_0+r)^2+(z_0-z)^2$, obtained by taking logarithmic differential of $k^2=4rr_0/(r_0+r)^2+(z_0-z)^2$.

Hence

$$M_{\theta_0}=\tfrac{1}{2}\,\pi i_0 a\int\limits_a^b\int\limits_{-l}^{l}\frac{2}{(rr_0)^{\frac{1}{2}}}\cdot\frac{(z_0-z)k}{(r_0+r)^2+(z_0-z)^2}\,dr\,dz.$$

$$=\tfrac{1}{2}\pi i_0 a\int_a^b\int_{-l}^{l}\frac{2\,(z_0-z)\,dr\,dz}{(rr_0)^{\frac{1}{2}}\{(r_0+r)^2+(z_0-z)^2\}}\cdot\frac{(4rr_0)^{\frac{1}{2}}}{\{(r_0+r)^2+(z_0-z)^2\}^{\frac{1}{2}}}$$

$$=2\pi i_0 a\odot,$$

$$\text{where }\odot=\int_a^b\int_{-l}^{l}\frac{(z_0-z)\,dr\,dz}{\{(r_0+r)^2+(z_0-z)^2\}^{\frac{3}{2}}}$$

$$=\int_a^b\left|\,\{(r_0+r)^2+(z_0-z)^2\}^{-\frac{1}{2}}\,\right|_{-l}^{l}\cdot dr.$$

$$=\int_a^b\Big[(r_0+r)^2+(z_0-l)^2\Big]^{-\frac{1}{2}}dr$$

$$-\int_a^b\Big[(r_0+r)^2+(z_0+l)^2\Big]^{-\frac{1}{2}}dr.$$

$$=\left|\,ln\,\frac{(r_0+r)^2+\sqrt{(r_0+r)^2+(z_0-l)^2}}{(r_0+r)^2+\sqrt{(r_0+r)^2+(z_0+l)^2}}\,\right|_a^b$$

$$=ln\,\frac{\{(r_0+b)+\sqrt{(r_0+b)^2+(z_0-l)^2}\}}{\{r_0+b)+\sqrt{(r_0+b)^2+(z_0+l)^2}\}}\cdot\frac{\{(r_0+a)+\sqrt{(r_0+a)^2+(z_0+l)^2}\}}{\{(r_0+a)+\sqrt{(r_0+a)^2+(z_0-l)^2}\}}$$

It can be proved very easily :

(*i*) $M_{\theta_0}(-z_0)=-M_{\theta_0}(z_0)$,

(*ii*) $M_{\theta_0}=0$, at any point in the central plane perpendicular to the axis.

(*iii*) $M_{\theta_0}=4\Delta b^{\cdot}{}_0/r_0{}^3$, when r_0 is very large and $\Delta=4\pi i_0 al$, the total current.

(*iv*) When z_0 is larger compared to l and r_0, the logarithmic function vanishes.

4. As a second approximation put

$$F=\frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2 k^2\right], \quad E=\frac{\pi}{2}\left[1-\left(\frac{1}{2}\right)^2 k^2\right].$$

$$\therefore M_{\theta_0}=\tfrac{1}{2}\pi i_0 a\int_a^b\int_{-l}^{l}\frac{2}{(rr_0)^{\frac{1}{2}}}\,\frac{\partial}{\partial z_0}\left[\left(\frac{2}{k}-k\right)\left(1+\frac{1}{4}k^2\right)-\frac{2}{k}\left(1-\frac{1}{4}k^2\right)\right]dr\,dz.$$

$$=\tfrac{1}{2}\pi i_0 a\int_a^b\int_{-l}^{l}\frac{2}{(rr_0)^{\frac{1}{2}}}\,\frac{\partial}{\partial z_0}\left(-\frac{1}{4}k^3\right)dr\,dz$$

$$=\tfrac{1}{2}\pi i_0 a\int_a^b\int_{-l}^{l}\frac{2}{(rr_0)^{\frac{1}{2}}}\cdot\left(-\frac{3}{4}k^2\,\frac{\partial k}{\partial z_0}\right)dr\,dz$$

$$=\tfrac{3}{4}\pi i_0 a\int_a^b\int_{-l}^{l}\frac{k^2}{(rr_0)^{\frac{1}{2}}}\cdot\frac{k\,(z_0-z)}{(r_0+r)^2+(z_0-z)^2}\,dr\,dz$$

$$=\tfrac{3}{4}\pi i_0 a\int_a^b\int_{-l}^{l}\frac{(z_0-z)\,dr\,dz}{(rr_0)^{\frac{1}{2}}\{(r_0+r)^2+(z_0-z)^2\}}\cdot\frac{(4rr_0)^{\frac{3}{2}}}{\{(r_0+r)^2+(z_0-z)^2\}^{\frac{3}{2}}}$$

$$=6\,\pi i_0 a\int_a^b\int_{-l}^{l}\frac{rr_0\,(z_0-z)\,dr\,dz}{\{(r_0+r)^2+(z_0-z)^2\}^{\frac{5}{2}}}\,.$$

$$=2\,\pi i_0 a\int_a^b\Big|\,rr_0\{(r_0+r)^2+(z_0-z)^2\}^{-\frac{3}{2}}\Big|_{-l}^{l}\cdot dr$$

$$=2\pi i_0 a\int_a^b\left[\frac{rr_0\,dr}{\{(r_0+r)^2+(z_0-l)^2\}^{\frac{3}{2}}}-\frac{rr_0\,dr}{\{(r_0+r)^2+(z_0+l)^2\}^{\frac{3}{2}}}\right]$$

Now
$$\int_a^b\frac{r\,dr}{\{(r_0+r)^2+(z_0-l)^2\}^{\frac{3}{2}}}=\int_a^b\frac{(r_0+r)\,dr}{\{(r_0+r)^2+(z_0-l)^2\}^{\frac{3}{2}}}$$

$$-\int_a^b\frac{r_0\,dr}{\{(r_0+r)^2+(z_0-l)^2\}^{\frac{3}{2}}}$$

$$=-\Big|\{(r_0+r)^2+(z_0-l)^2\}^{-\frac{1}{2}}\Big|_a^b$$

$$-r_0\int_{r=a}^b\frac{d\,(r_0+r)}{\{(r_0+r)^2+(z_0-l)^2\}^{\frac{3}{2}}}$$

Again
$$\int\frac{d\,(r_0+r)}{\{(r_0+r)^2+(z-l)^2\}^{\frac{3}{2}}}=\frac{r_0+r}{(z_0-l)^2\{(r_0+r)^2+(z_0-l)^2\}^{\frac{1}{2}}}$$

Hence

$$M_{\theta_0}=2\pi i_0 ar_0\left[-[(r_0+r)^2+(z_0-l)^2\}^{-\frac{1}{2}}\right.$$

$$-\frac{r_0\,(r_0+r)}{(z_0-l)^2\{(r_0+r)^2+(z_0-l)^2\}^{\frac{1}{2}}}$$

$$\left.+\{(r_0+r)^2+(z_0+l)^2\}^{-\frac{1}{2}}+\frac{r_0\,(r_0+r)}{(z_0+l)^2\{(r_0+r)^2+(z_0+l)^2\}^{\frac{1}{2}}}\right]_{r=a}^b$$

$$=2\pi i_0 ar_0\Big|\frac{r_0^2+rr_0+(z_0+l)^2}{(z_0+l)^2\{(r_0+r)^2+(z_0+l)^2\}^{\frac{1}{2}}}$$

$$-\frac{r_0^2+rr_0+(z_0-l)^2}{(z_0-l)^2\{(r_0+r)^2+(z_0-l)^2\}^{\frac{1}{2}}}\Big|_{r=a}^b$$

5. The value of M_{θ_0} can be expressed in terms of an integral equation involving Bessel's functions of the first type. Thus suppose

$$W=\int_0^{2\pi}\int_{-l}^{l}\int_a^b \frac{\cos I\, dr\, dz\, d I}{\sqrt{r_0^2+r^2+(z_0-z)^2-2rr_0 \cos I}}$$

Then since[1] $\int_0^\infty e^{-\lambda z} J_0(\lambda R)\, d\lambda = \frac{1}{(z^2+R^2)^{\frac{1}{2}}}$, we get

$$W=\int_0^\infty\int_0^{2\pi}\int_{-l}^{l}\int_a^b e^{\pm\lambda(z_0-z)} \cos I\, J_0(\lambda R)\, dr\, dz\, dI\, d\lambda,$$

the upper or lower sign being taken according as z_0-z is negative or positive, where $R^2=r_0^2+r^2-2rr_0 \cos I$.

Now[2] $J_0(\lambda R)=J_0(\lambda r_0)\, J_0(\lambda r)+2\sum_1^\infty J_s(\lambda r_0)\, J_s(\lambda r) \cos s\, I.$

Whence

$$W=\int_0^\infty d\lambda \int_{-l}^{l} e^{\pm\lambda(z_0-z)} dz \int_a^b dr \int_0^{2\pi} \Big\{ J_0(\lambda r_0)\, J_0(\lambda r)$$

$$+2\sum_1^\infty J_s(\lambda r_0)\, J_s(\lambda r) \cos s\, I \Big\} \cos I\, d\, I.$$

Now $\int_0^{2\pi} \cos I\, dI=0$, $\int_0^{2\pi} \cos s\, I \cos I\, d\, I= \begin{cases} 0, & (s\neq 1) \\ \pi, & (s=1). \end{cases}$

Hence $W=2\pi \int_0^\infty e^{\pm\lambda(z_0-z)} d\lambda \int_{-l}^{l} dz \int_a^b J_1(\lambda r_0)\, J_1(\lambda r)\, dr.$

[1] Gray and Mathews, 'Bessel functions.'

[2] Gray and Mathews, l.c.

Again since $J_0'(z) = -J_1(z)$, we have

$$\frac{d}{dr} J_0(\lambda r) = -\lambda J_1(\lambda r)$$

$$\therefore W = -2\pi \int_0^{\infty} e^{\pm\lambda(z_0 - z)} d\lambda \int_{-l}^{l} dz \left[J_1(\lambda r_0) \cdot \frac{1}{\lambda} J_0(\lambda r) \right]_{r=a}^{b}$$

$$= -2\pi \int_{-l}^{l} \int_0^{\infty} e^{\pm\lambda(z_0 - z)} J_1(\lambda r_0) \{ J_0(\lambda b) - J_0(\lambda a) \} \frac{dz\, d\lambda}{\lambda}.$$

$$= -2\pi \int_0^{\infty} e^{\pm\lambda z_0} (e^{\lambda l} - e^{-\lambda l}) J_1(\lambda r_0) J_0(\lambda b) \frac{d\lambda}{\lambda^2}.$$

$$+ 2\pi \int_0^{\infty} e^{\pm\lambda z_0} (e^{\lambda l} - e^{-\lambda l}) J_1(\lambda r_0) J_0(\lambda a) \frac{d\lambda}{\lambda^2},$$

Now $M_{\theta_0} = i_0 a \dfrac{\partial}{\partial z_0} W$

$$= \mp 2\pi i_0 a \left[\int_0^{\infty} e^{\pm\lambda z_0} (e^{\lambda l} - e^{-\lambda l}) J_1(\lambda r_0) J_0(\lambda b) \frac{d\lambda}{\lambda} \right.$$

$$\left. - \int_0^{\infty} e^{\pm\lambda z_0} (e^{\lambda l} - e^{-\lambda l}) J_1(\lambda r_0) J_0(\lambda a) \frac{d\lambda}{\lambda} \right].$$

The *minus* sign before the right-hand side corresponds to $+\lambda z_0$ and *plus* sign corresponds to $-\lambda z_0$; $+\lambda z_0$ is taken for $z_0 < l$ and $-\lambda z_0$ for $z_0 > l$.

We find

$$M_{\theta_0} = -2\pi i_0 a \int_0^\infty e^{\pm\lambda(z_0+l)} J_1(\lambda r_0) J_0(\lambda b) \frac{d\lambda}{\lambda}$$

$$+2\pi i_0 a \int_0^\infty e^{\pm\lambda(z_0-l)} J_1(\lambda r_0) J_0(\lambda b) \frac{d\lambda}{\lambda}$$

$$+2\pi i_0 a \int_0^\infty e^{\pm\lambda(z_0+l)} J_1(\lambda r_0) J_0(\lambda a) \frac{d\lambda}{\lambda}$$

$$-2\pi i_0 a \int_0^\infty e^{\pm\lambda(z_0-l)} J_1(\lambda r_0) J_0(\lambda a) \frac{d\lambda}{\lambda}.$$

As before the $\pm\lambda$ is taken according as $z_0 <$ or $> l$.

These are well-known integrals. We have met with this type of integrals in Hydrodynamics as the expression for velocity potential for sources distributed with uniform density over the plane area contained by a circle,[1] also from analogy the gravitational potential produced at any outside-point due to a thin disc of matter of uniform surface density has got the same value.[2]

Lastly I wish to express my thanks to Prof. M. N. Saha, D.Sc., who suggested the problem to me for his encouragement in this direction.

[1] Lamb's 'Hydrodynamics,' 4th edition, p. 131.
[2] Gray. 'Phil. Mag., August, 1919,' p. 203.

4

Ripples of Finite Amplitude on a Viscous Liquid

BY

J. C. Kamesvara Rav, M.Sc.

In a paper published in this bulletin,[1] I showed the change of form of waves of finite amplitude, as the wave length increases from those of short ripples to those of large gravity waves, without taking into consideration the effect of viscosity, which however, is not a negligible factor, as it tends to damp the amplitude, which in its turn affects the form of the wave.

The effect of viscosity on waves on the surface of liquids first received the attention of Stokes,[2] who by employing the dissipation function found the modulus of decay to vary as ν^{-1}. Tait[3] discussed the effect on short ripples and showed that it is more prominent in the case of shorter ripples. Harrison[4] found for superposed liquids, the modulus of decay to vary as $\nu^{-\frac{1}{2}}$. Basset[5] extended the case to liquids of finite depth. Recently Watson[6] made some experimental investigations to find the viscosity of liquids by taking observations of the decay of the amplitude of surface waves. He, however worked only with small amplitudes. Taking the effect of finiteness of the amplitude, we can proceed as follows—

The motion is supposed to be confined to two dimensions. The axis of X is drawn in the direction of propagation and the axis of Y is drawn vertically upwards.

[1] Bull. of the Cal. Math. Soc. Vol. XI, p. 173 (1920). See also Proc. Ind. Ass. f. Cult. of Science. Vol. VI, p. 175 (1921).

[2] Camb. Trans. t. ix, p. 8, 1855 or papers Vol. III, p. 1.

[3] Proc. Roy. Soc. Edin. Vol. XVII, p 110 (1890) or Scientific papers Vol. II. p. 318.

[4] Proc. Lond. Math. Soc. (2) Vol. VI, p. 396 and Vol. VII, p. 107 (1908).

[5] Hydrodynamics, Vol. II, §§. 520-522 (1888).

[6] Bhys. Rev. Vol. VII, p. 226 (1916).

The well known equations for the motion of a viscous fluid are

$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$

In the present case these reduce to

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \quad \ldots \quad (1)$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 v \quad \ldots \quad (2)$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \ldots \quad (3)$$

These equations are satisfied by

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y} \\ v &= -\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}, \end{aligned}\right\} \quad \ldots \quad (4)$$

and the pressure equation at the surface

$$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - gy - \frac{1}{2}(u^2 + v^2) \quad \ldots \quad (5)$$

provided that

$$\nabla_1^2 \phi = 0, \text{ and } \frac{\partial \psi}{\partial t} = \nu\nabla^2\psi, \quad \ldots \quad (6)$$

where $\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

The condition of 'no motion' at the bottom is given by

$$-\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} = 0, \text{ when } y = -h. \quad \ldots \quad (7)$$

Solutions of (6) and (7) are given by

$$\phi=\sum_{s=n}^{s=1} A_s e^{sat} \cosh sk(y+h)\cos skx$$

and

$$\psi=\sum_{s=n}^{s=1} B_s e^{sat} \sinh sm(y+h)\sin skx$$

where

$$m^2=k^2+\frac{a}{\nu}. \qquad \dots \quad (8)$$

The boundary conditions supply enough equations to determine the constants $A_2, A_3 \dots B_2, B_3 \dots$ and a, A_1 and B_1 (which are related to each other) remaining arbitrary.

For our purpose it is sufficient to take only three terms, and we have

$$\phi=A_1 e^{at} \cosh k(y+h)\cos kx+A_2 e^{2at} \cosh 2k(y+h)\cos 2kx$$
$$+A_3 e^{3at} \cosh 3k(y+h)\cos 3kx, \qquad \dots \quad (9)$$

and

$$\psi=B_1 e^{at} \sinh m(y+h)\sin kx+B_2 e^{2at} \sinh 2m(y+h)\sin 2kx$$
$$+B_3 e^{3at} \sinh 3m(y+h)\sin 3kx. \qquad \dots \quad (10)$$

Substituting these values in equations (4), we get

$$u=\{kA_1\cosh k(y+h)-mB_1\cosh m(y+h)\}e^{at}\sin kx$$
$$+2\{kA_2\cosh 2k(y+h)-mB_2\cosh 2m(y+h)\}e^{2at} \sin 2kx$$
$$+3\{kA_3\cosh 3k(y+h)-mB_3\cosh 3m(y+h)\}e^{3at} \sin 3kx. \qquad (11)$$

$$v=-k\{A_1\sinh k(y+h)-B_1\sinh m(y+h)\}e^{at} \cos kx$$
$$-2k\{A_2\sinh 2k(y+h)-B_2\sinh 2m(y+h)\}e^{2at} \cos 2kx$$
$$-3k\{A_3\sinh 3k(y+h)-B_3\sinh 3m(y+h)\}e^{3at} \cos 3kx. \quad \dots \quad (12)$$

If η denotes the elevation of the free surface and if the origin be taken in the undisturbed level of the liquid we have

$$v=\frac{\partial\eta}{\partial t}, \quad \text{and}$$

$$\eta=-\frac{k}{a}\Big[(A_1\sinh kh-B_1\sinh mh)e^{at}\cos kx$$
$$+(A_2\sinh 2kh-B_2\sinh 2mh)e^{2at} \cos 2kx$$
$$+(A_3\sinh 3kh-B_3\sinh 3mh)e^{3at} \cos 3kx\Big]. \qquad \dots \quad (13)$$

The stress conditions at the surface ($\eta=0$), give

$$p_{yy}=T\frac{\partial^2\eta}{\partial x^2}, \qquad \dots \quad (14)$$

and
$$p_{xy}=0, \qquad \dots \quad (15)$$

where T is the surface tension of the liquid. The curvature is supposed to be small.

But

$$p_{yy}=-p+2\mu\frac{\partial v}{\partial y} \qquad \dots \quad (16)$$

$$p_{xy}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right), \qquad \dots \quad (17)$$

where $\mu=\rho\nu$.

Equations (17) and (16) together give the values of the constants A_2, A_3..., B_2, B_3 etc. and a. Equation (17), gives on substitution

$$\{2A_1k^2\sinh kh-B_1(k^2+m^2)\sinh mh\}e^{at}\sin kx+4\{2A_2k^2\sinh 2kh$$
$$-B_2(k^2+m^2)\sinh 2mh\}e^{2at}\sin 2kx+9\{2A_3k^2\sinh 3kh$$
$$-B_3(k^2+m^2)\sinh 3mh)\}e^{3at}\sin 3kx=0.$$

Equating the coefficients of $\sin kx$, $\sin 2kx$ and $\sin 3kx$ to zero, we get,

$$\frac{B_1}{A_1}=\frac{2k^2\sinh kh}{(k^2+m^2)\sinh mh} \qquad \dots \quad (18)$$

$$\frac{B_2}{A_2}=\frac{2k^2\sinh 2kh}{(k^2+m^2)\sinh 2mh} \qquad \dots \quad (19)$$

$$\frac{B_3}{A_3}=\frac{2k^2\sinh 3kh}{(k^2+m^2)\sinh 3mh} \qquad \dots \quad (20)$$

Equations (16) and (14) to gether give, after substituting the values of η, u and v from (12), (13) and (14)

$$0=\frac{k}{a}\left(g+\frac{T}{\rho}k^2\right)\left(A_1\sinh kh-B_1\sinh mh\right)e^{at}\cos kx$$

$$+\frac{k}{a}\left(g+\frac{4T}{\rho}k^2\right)\left(A_2\sinh 2kh-B_2\sinh 2mh\right)e^{2at}\cos 2kx$$

$$+\frac{k}{a}\left(g+\frac{9T}{\rho}k^2\right)\left(A_3\sinh 3kh-B_3\sinh 3mh\right)e^{3at}\cos 3kx$$

$$+A_1 a e^{at}\cos kx\cosh k(\eta+h)+2A_2 a e^{2at}\cos 2kx\cosh 2k(\eta+h)$$

$$+3A_3 a e^{3at}\cos 3kx\cosh 3k(\eta+h)+2\nu k\{A_1 k\cosh k(\eta+h)$$

$$-mB_1\cosh m(\eta+h)\}e^{at}\cos kx+4\nu k\{A_2 k\cosh 2k(\eta+h)$$

$$-mB_2\cosh 2m(\eta+h)\}e^{2at}\cos 2kx+6\nu k\{A_3 k\cosh 3k(\eta+h)$$

$$-mB_3\cosh 3m(\eta+h)\}e^{3at}\cos 3kx$$

$$-\frac{1}{2}\Big[\{kA_1\cosh k(\eta+h)-mB_1\cosh m(\eta+h)\}e^{at}\sin kx$$

$$+2\{kA_2\cosh 2k(\eta+h)-mB_2\cosh 2m(\eta+h)\}e^{2at}\sin 2kx$$

$$+3\{kA_3\cosh 3k(\eta+h)-mB_3\cosh 3m(\eta+h)\}e^{3at}\sin 3kx\Big]^2$$

$$-\frac{k^2}{2}\Big[\{A_1\sinh k(\eta+h)-B_1\sinh m(\eta+h)\}e^{at}\cos kx$$

$$+2\{A_2\sinh 2k(\eta+h)-B_2\sinh 2m(\eta+h)\}e^{2at}\cos 2kx$$

$$+3\{A_3\sinh 3k(\eta+h)-B_3\sinh 3m(\eta+h)\}e^{3at}\cos 3kx\Big]^2.$$

Expanding the hyperbolic functions in powers of $(\eta+h)$ and substituting the value of η given by (13) in the above equation and equating the coefficients of $\cos kx$, $\cos 2kx$ and $\cos 3kx$ to zero, we get,

$$\frac{k}{a}\left(g+\frac{Tk^2}{\rho}\right)(A_1\sinh kh-B_1\sinh mh)+A_1 a\cosh kh$$
$$+2\nu k(A_1 k\cosh kh-B_1 m\cosh mh)=0, \quad \dots \quad (21)$$

$$\frac{k}{a}\left(g+\frac{4T}{\rho}k^2\right)(A_2\sinh 2kh-B_2\sinh 2mh)+2A_2 a\cosh 2kh$$

$$+\frac{1}{2}k^3A_1\sinh kh(A_1\sinh kh-B_1\sinh mh)+4\nu k(A_2 k\cosh 2kh$$

$$-B_2 m\cosh 2mh)+\nu k^6/a\cdot A_1(A_1\sinh kh-B_1\sinh mh)\sinh kh$$

$$+\nu/a\cdot k^3m^2B_1(A_1\sinh kh-B_1\sinh mh)\sinh mh$$

$$-\frac{1}{2}(kA_1\cosh kh-mB_1\cosh mh)^2-\frac{k^2}{2}(A_1\sinh kh-B_1\sinh mh)^2=0, \quad (2$$

$$\frac{k}{a}\left(g+\frac{9Tk^2}{\rho}\right)(A_3\sinh 3kh-B_3\sinh 3mh)+3A_3 a\cosh 3kh$$

$$+6\nu k(A_3 k\cosh 3kh-B_3 m\cosh 3mh)$$

$$+\frac{1}{8}\ A_1\frac{k^4}{a}(A_1\sinh kh-B_1\sinh mh)^2\cosh kh$$

$$-\frac{1}{2}\ A_1^2k(A_2\sinh 2kh-B_2\sinh 2mh)\sinh kh$$

$$-2A_2k^2(A_1\sinh kh-B_1\sinh mh)\sinh 2kh$$

$$+2\nu k\left[\ \frac{A_1}{8}\ \frac{k^5}{a^2}(A_1\sinh kh-B_1\sinh mh)^2\cosh kh\right.$$

$$-\frac{1}{2}\ A_1\frac{k^3}{a}(A_2\sinh 2kh-B_2\sinh 2mh)\sinh kh$$

$$\left.-2A_2\ \frac{k^3}{a}(A_1\sinh kh-B_1\sinh mh)\ \sinh 2kh\ \right]$$

$$+2\nu k\left[\ -\ \frac{B_1}{8}\ \ \frac{k^2m^3}{a^2}\ (A_1\sinh kh-B_1\sinh mh)^2\cosh mh\right.$$

$$+\frac{1}{2}\ B_1\frac{km^2}{a}\ (A_2\sinh 2kh-B_2\sinh 2mh)\sinh mh$$

$$\left.+2B_2\frac{km^2}{a}\ (A_1\sinh kh-B_1\sinh mh)\sinh 2mh\ \right]$$

$$-(kA_1\cosh kh-mB_1\cosh mh)(kA_2\cosh 2kh-mB_2\cosh 2mh)$$

$$-k^2(A_1\sinh kh-B_1\sinh mh)(A_2\sinh 2kh-B_2\sinh 2mh)=0. \quad \ldots \quad (23)$$

Substituting the value of B_1/A_1 as given by (18) in equation (21), we get with the help of (8)

$$-\frac{k}{a}\left(g+\frac{Tk^2}{\rho}\right)\frac{\sinh kh}{2k^2\nu+a}-(a+2\nu k^2)\cosh kh=0,$$

neglecting squares of ν.

The above equation gives

$$a=-2\nu k^2\mp ik(g/k+Tk/\rho)^{\frac{1}{2}}$$

$$=-2\nu k^2\mp ikc \qquad \ldots \quad (24)$$

where c is written for $(g/k+Tk/\rho)$, the velocity of the wave in absence of friction.

Equation (22) becomes on the substitution of the values of B_1 and B_2 from (18) and (19) and with the help of (8)

$$\frac{A_2k}{a}\left(g+\frac{4Tk^2}{\rho}\right)\left(\frac{a}{2k^2\nu+a}\right)\sinh 2kh+2A_2a\cosh 2kh$$

$$+\frac{1}{2}\,k^3A_1^2\,\frac{a}{2k^2\nu+a}\cdot\sinh^2 kh+4k\nu\,A_2\left\{k\cosh 2kh\right.$$

$$\left.-\frac{2k^2\nu}{2k^2\nu+a}\cdot\frac{\sinh 2kh}{\tanh 2mh}\right\}+\frac{\nu k^5A_1^2}{2k^2\nu+a}\,\sinh^2 kh$$

$$\frac{2k^5m^2\nu^2A_1^2}{(2k^2\nu+a)^2}\,\sinh^2 kh-\frac{1}{2}\,k^2A_1^2\cosh 2kh$$

$$-\frac{1}{2}\,A_1^2\,\frac{2k^2\nu\sinh kh}{(2k^2\nu+a)\sinh mh}\left(k^2\cosh 2mh+\frac{a}{\nu}\cosh^2 mh\right)=0.$$

From this we get, neglecting squares of ν

$$A_2=-\frac{\frac{1}{2}k^3A_1^2a\sinh^2 kh+\nu k^6A_1^2\sinh^2 kh+}{k\left(g+\frac{4Tk^2}{\rho}\right)\sinh 2kh+2a\cosh 2kh(2k^2\nu+a)-4k\nu[k(2k^2\nu+a)\cosh 2kh-2\nu k^3\sinh 2kh\coth 2mh]}\quad\ldots\quad(25)$$

Equation (23) becomes

$$\frac{A_3k}{a}\left(g+\frac{9Tk^2}{\rho}\right)\frac{a\cdot\sinh 3kh}{2k^2\nu+a}+3A_3a\cosh 3kh+6A_3\nu k\left[k\cosh 3kh\right.$$

$$\left.-\frac{2mk^2\nu}{2k^2\nu+a}\cdot\frac{\sinh 3kh}{\coth 3mh}\right]+\frac{1}{8}\,A_1^3\,\frac{k^4\cdot a}{(2k^2\nu+a)^2}\,\sinh^2 kh\cosh kh$$

$$-\frac{5}{2}\,A_1A_2\,\frac{ak^2}{2k^2\nu+a}\,\sinh 2kh\sinh kh.$$

$$+2\nu k\left\{\frac{2k^2\nu}{2k^2\nu+a}\sinh kh\coth mh\left[\frac{A_1^3}{8}\quad\frac{k^5}{(2k^2\nu+a)^2}\quad\sinh^2 kh\right.\right.$$

$$\left.-\frac{1}{2}\,A_1A_2\,\frac{km^2}{2k^2\nu+a}\,\sinh 2kh\tanh mh\right]$$

$$\left.+2A_1A_2km^2\,\frac{2k^2\nu}{2k^2\nu+a}\,\sinh 2kh\sinh kh\right\}.$$

$$+A_1A_2\left(k\cosh kh-\frac{2mk^2\nu}{2k^2\nu+a}\sinh kh\coth mh\right)\left(k\cosh 2kh\right.$$

$$\left.-\frac{2mk^2\nu}{2k^2v+a}\sinh 2kh\tanh 2mh\right)-A_1A_2k^2\cdot\frac{a^2}{(2k^2\nu+a)^2}$$

$$\sinh kh\sinh 2kh+\text{etc.}=0.$$

This, on further simplification becomes

$$A_3 k\left(g+\frac{9Tk^2}{\rho}\right)\frac{\sinh 3kh}{2k^2\nu+a}+3A_3 a\cosh 3kh+6A_3\nu k\left[k\cosh 3kh\right.$$

$$\left.-\frac{2mk^2\nu}{2k^2\nu+a}\cdot\frac{\sinh 3kh}{\coth 3mh}\right]+\frac{A_1^3}{4}\quad\frac{k^4 a}{2k^2\nu+a}\sinh^2 kh\cosh kh$$

$$-\frac{5}{2}A_1A_2\sinh kh\sinh 2kh=0.$$

From this,

$$A_3=\frac{\frac{5}{2}A_1A_2k(2k^2\nu+a)^2\sinh kh\sinh 2kh-\frac{A_1^3}{4}k^4c^2\sinh^2 kh\cosh kh}{k\left(g+\frac{9Tk^2}{\rho}\right)\sinh 3kh+3a(2k^2\nu+a)\cosh 3kh+6\nu k[(2k^2\nu+a)\cosh 3kh-2mk^2\nu\sinh 3kh\tanh 3mh]}\quad\ldots\quad(26)$$

With the help of relations (18), (19) and (20), the equation of the wave surface can be written as

$$\eta=-\frac{k}{2k^2\nu+a}\left[A_1e^{at}\sinh kh\cos kx+A_2e^{2at}\sinh 2kh\cos 2kx\right.$$

$$\left.+A_3e^{3at}\sinh 3kh\cos 3kx\right].$$

Substituting the value of a, given by (24), we get

$$\eta=-\frac{k}{2k^2\nu+a}\left[A_1e^{-2k^2\nu t}e^{ikct}\sinh kh\cos kx\right.$$

$$\left.+A_2e^{-4k^2\nu t}e^{2ikct}\sinh 2kh\cos 2kx+A_3e^{-6k^2\nu t}e^{3ikct}\sinh 3kh\cos 3kx\right]$$

Substituting the values of A_2 and A_3 given by (25) and (26) we get,

$$\eta=-\frac{k}{2k^2\nu+a}\left[A_1e^{-2k^2\nu t}e^{ikct}\sinh kh\right.$$

$$-\frac{\frac{1}{2}k^3A_1^2a\sinh^2 kh+\nu k^5A_1^2\sinh^2 kh+}{k\left(g+\frac{4Tk^2}{\rho}\right)\sinh 2kh+2(2k^2\nu+a)a\cosh 2kh}$$

$$\times e^{-4k^2\nu t}e^{2ikct}\sinh 2kh$$

$$+\frac{\frac{5k^2}{2}A_1A_2(2k^2\nu+a)^2\sinh kh\sinh 2kh-\frac{A_1^3}{4}k^4c^2\sinh^2 kh\cosh kh}{k\left(g+\frac{4Tk^2}{\rho}\right)\sinh 3kh+3(2k^2\nu+a)a\cosh 3kh}.$$

Putting $$-\frac{kA_1\sinh kh}{2k^2\nu+a}=a, \text{ we get}$$

$$\eta=ae^{-2\nu k^2 t}e^{ikct}\cos kx$$

$$+\frac{\left(\frac{1}{2}k^3c^2+\nu k^3c^2\right)a^2\cos 2kx}{k\left(g+\frac{4Tk^2}{\rho}\right)\sinh 2kh-2kac\cosh 2kh}\cdot e^{-4\nu k^2 t}e^{2ikct}\sinh 2kh$$

$$+\frac{\frac{5}{2}aA_2c^2k^2\sinh 2kh-\frac{a^3}{8}k^4c^2\coth kh}{\left(g+\frac{9Tk^2}{\rho}\right)\sinh 3kh-3kca\cosh 3kh}e^{-6\nu k^2 t}e^{3ikct}\sinh 3kh\cos 3kx.$$

In real quantities this can be written as

$$\eta=ae^{-2\nu k^2 t}\cos kct\cos kx$$

$$+\frac{\left(\frac{1}{2}k^2c^2+\nu k^2c^2\right)e^{-4\nu k^2 t}\sinh 2kh\cos 2kct\cos 2kx}{\left(g+\frac{4Tk^2}{\rho}\right)\sinh 2h-2kc^2\cosh 2kh}$$

$$+\frac{\frac{5}{2}aA_2k^2c^2\sinh 2kh-\frac{a^3}{8}k^4c^2\coth kh}{\left(g+\frac{9Tk^2}{\rho}\right)\sinh 3kh-3kc^2\cosh 3kh}$$

$$\times e^{-6\nu k^2 t}\sinh 3kh\cos 3kct\cos 3kx. \qquad \dots \quad (27)$$

For infinite depth, this becomes finally,

$$\eta=ae^{-2\nu k^2 t}\cos kct\cos kx$$

$$+\frac{\left(\frac{1}{2}k^2c^2+\nu k^2c^2\right)a^2}{\left(g+\frac{4Tk^2}{\rho}\right)-2kc^2}e^{-4\nu k^2 t}\cos 2kct\cos 2kx$$

$$+\frac{\frac{5}{2}aa'k^2c^2-\frac{a^3}{8}k^4c^2}{\left(g+\frac{9Tk^2}{\rho}\right)-3kc^2}e^{-6\nu k^2 t}\cos 3kct\cos 3kx \qquad \dots \quad (28)$$

where a' is given by

$$\frac{\frac{1}{2} k^2 c^2 + \nu k^2 c^2}{\left(g + \frac{4Tk^2}{\rho} \right) - 2kc^2} .$$

Equation (28) shows that the effect of viscosity is greater on the second term than on the first ; hence the viscosity has an effect on the form of the wave similar to that of divergence. The above solution also shows that the viscosity effects the form of small ripples more than those of large gravity waves. A similar affect has also been observed experimentally. In the case of long waves, the division of waves into two crests, extends over many wave lengths, while in the case of short ripples no division is noticed. The author hopes to make a more detailed experimental verification in the near future.

Geometrical Investigations on the Correspondences between a Right-angled Triangle, a Three-right-angled Quadrilateral and a Rectangular Pentagon in Hyperbolic Geometry.

BY

S. Mukhopadhyaya.

Theorem 1.

ABC is a triangle right-angled at C in a given hyperbolic plane. AU is parallel to CB and DV is parallel to AB, where D is a point in AC produced and the angle ADV is a right angle. EF is the common perpendicular to AU and DV, meeting AU in E and DV in F.

Then is AE equal to AB and DF equal to CB. See Fig. 1.

Proof.

With a view to perspicuity the proof will be divided into several distinct parts.

(1)

Let G be the middle point of CD and H of EF. Produce CB to X and AB to Y Join GH and produce it to Z.

Then GZ is parallel to CX. See Fig. 1.

As G is the middle point of CD which is common perpendicular to CX and DV, a parallel GW to VD will in the opposite sense WG be parallel to CX and therefore to AU. Hence WG passes through the middle point H of the common perpendicular EF to AU and DV.

(2)

From AU cut off AE′ equal to AB and from DV cut off DF′ equal to CB.

Then the angles AE′F′ and DF′E′ are equal. See Fig. 2.

Let p denote the line which is common parallel[1] to CX and AY and consequently also common parallel to AU and DV. Join BE$'$ and BF$'$. The bisector of the angle BAE$'$ which bisects BE$'$ at right angles is perpendicular to p. The perpendicular bisector of CD which bisects BF$'$ at right angles is also perpendicular to p. It follows from Bolyai's Theorem that the perpendicular to E$'$F$'$ through its middle point H$'$ is also perpendicular to p. It follows from considerations of symmetry that the angles AE$'$F$'$ and DF$'$E$'$ are equal.

(3)

Produce AD to K making DK equal to AC, so that G the middle point of CD is also the middle point of AK. Join KF$'$. From the congruence of the triangles KDF$'$ and ACB, KF$'$ is equal to AB and parallel to CX.

Then GH$'$ joining the middle points of AK and E$'$F$'$ is parallel to CX. See Fig. 3.

Through G draw the parallel GZ to AU. Produce KF$'$ to T. Then GZ is also parallel to CX and KT. Draw perpendiculars AM, E$'$N, KP, F$'$Q on GZ. Then AM is equal to KP because AG is equal to GK. Consequently the angle of parallelism MAU is equal to the angle of parallelism PKT. Also AE$'$ is equal to KF$'$ as each of them is equal to AB. Therefore the figures AMNE$'$ and KPQF$'$, are congruent, so that E$'$N is equal to F$'$Q. Hence the line GZ passes through H$'$ the middle point of E$'$F$'$. See Fig. 3.

(4)

It follows from (1), (2), (3) that EF must coincide with E$'$F$'$.

For if EF do not coincide with E$'$F$'$, it follows from (2) that HH$'$ is perpendicular to EF. But it follows from (1) and (3) that HH$'$ is parallel to AE and therefore cannot be perpendicular to EF.

Thus Theorem 1 is completely proved.

Corollary 1, Theorem 1.

If a and b denote two sides of a right-angled triangle and c the hypotenuse and if λ, μ denote the angles opposite the sides a, b, then a three-right-angled quadrilateral can always be constructed of which the fourth angle is β and the sides reckoned in order from this side are l, a, m', c,

[1] Hilbert in his *Grundlagen* gives an elegant construction of the common parallel which is independent of the *Postulate of Archimedes*, reproduced by Carslaw in his Non-Euclidean Geometry, p. 55.

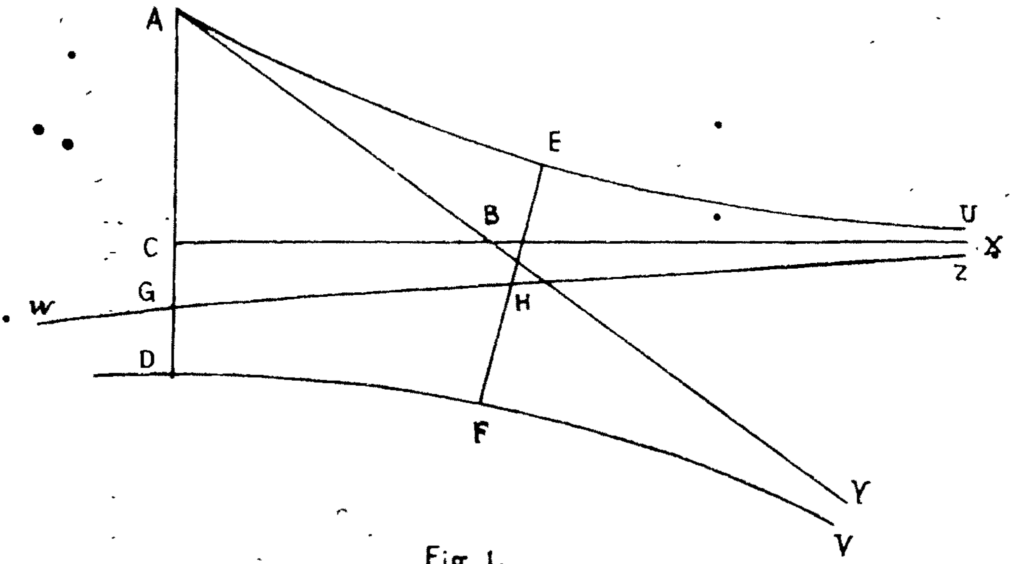


Fig 1.

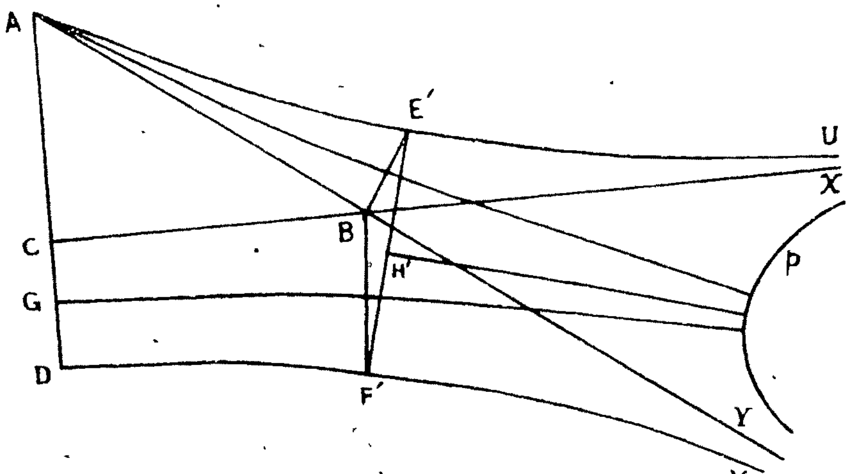


Fig 2.

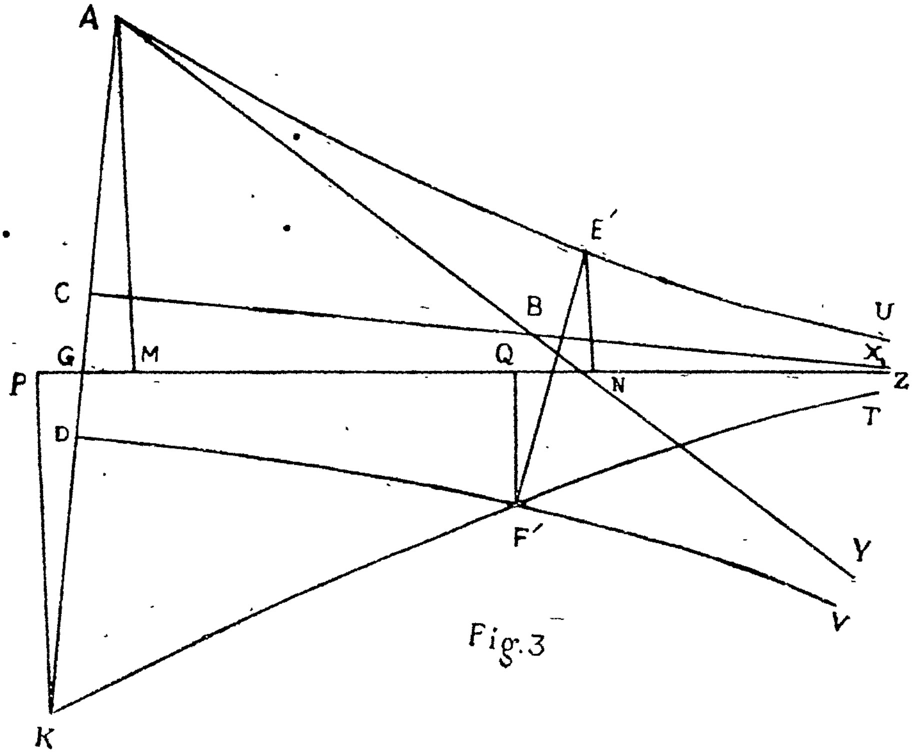


Fig. 3

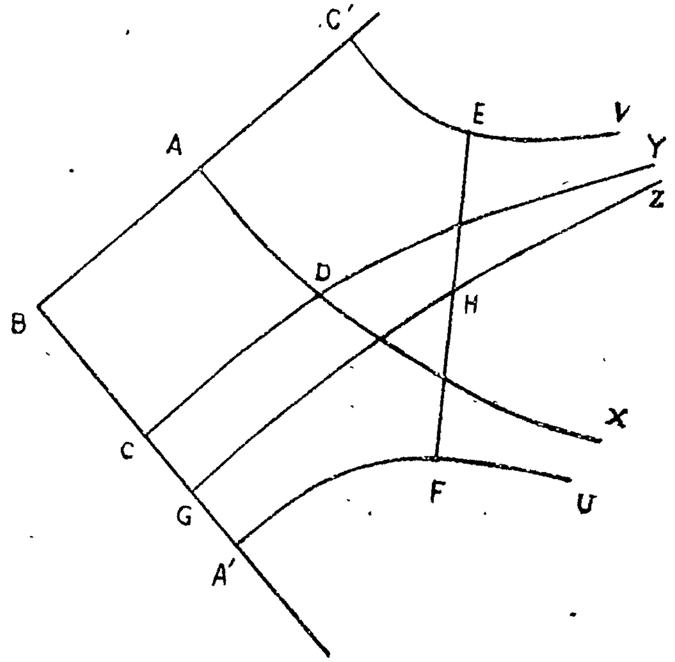


Fig. 4

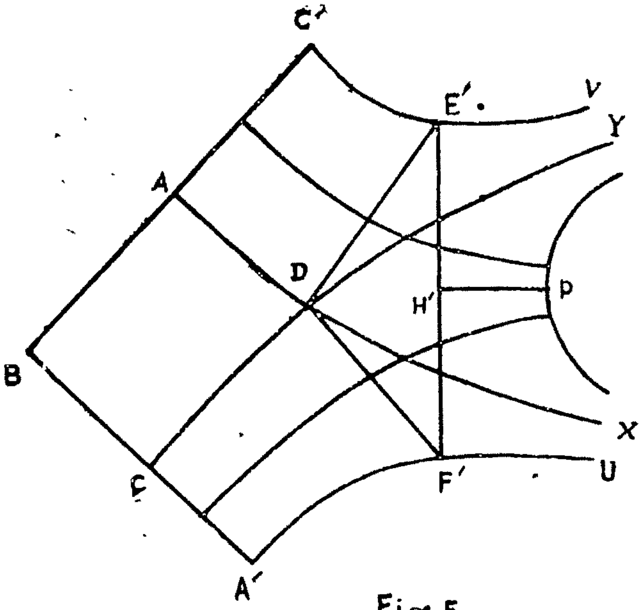


Fig. 5.

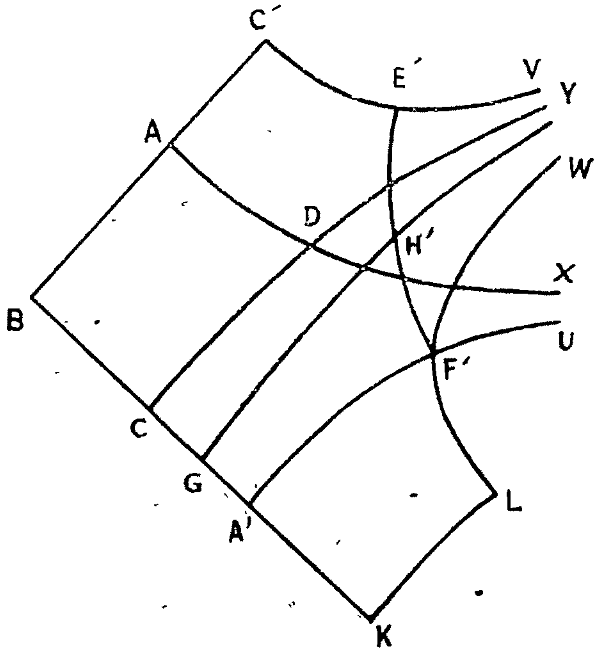


Fig. 6.

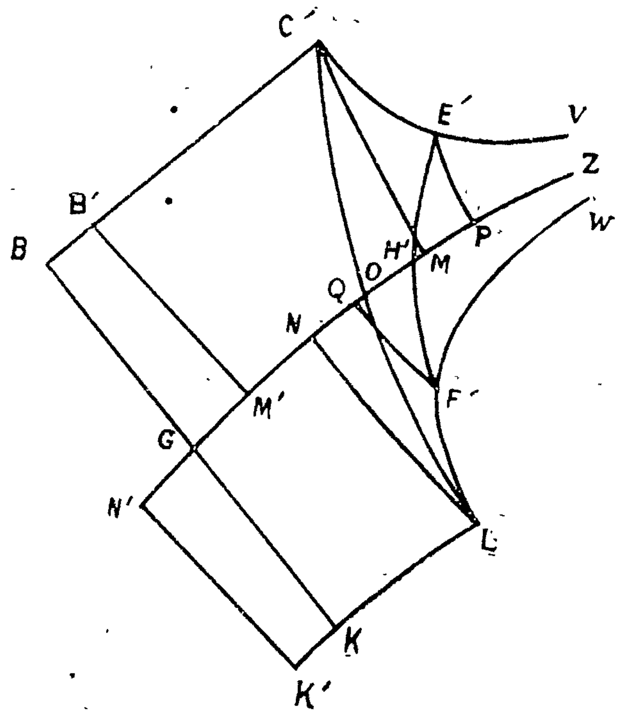


Fig. 7.

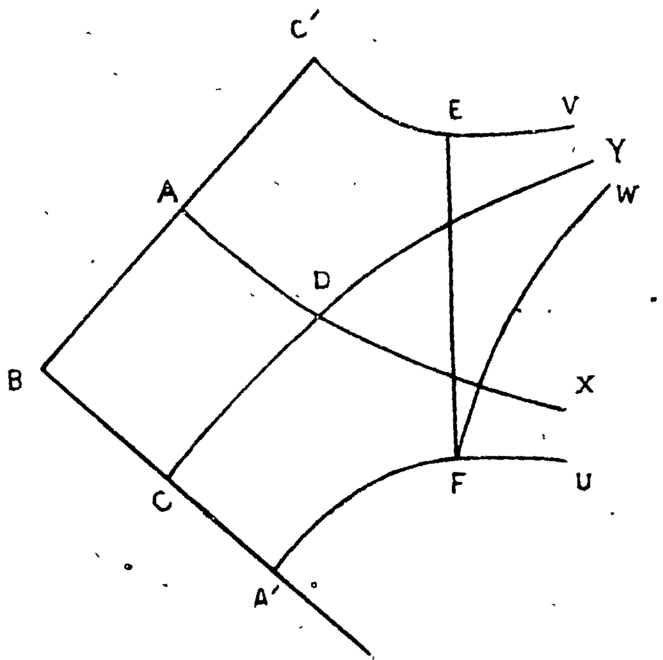


Fig. 8

NOTATION:—The angles of parallelism corresponding to lengths a, b, c, l, m are represented by the corresponding Greek letters α, β, γ λ, μ, and a', b', c', l', m' are lengths corresponding to angles of parallelism $\frac{\pi}{2}-\alpha$, $\frac{\pi}{2}-\beta$, $\frac{\pi}{2}-\gamma$, $\frac{\pi}{2}-\lambda$, $\frac{\pi}{2}-\mu$, and are called lengths complementary to a, b, c, l, m, n.

Theorem 1 gives the three-right-angled quadrilateral ADFE corresponding to the right-angled-triangle ABC. The angle DAE is β, being the angle of parallelism corresponding to distance AC which is b. The distance AD is l as it corresponds to the angle of parallelism CAB which is λ, and DF is equal to CB which is a. Also AE is equal to AB which is c.

The angle of parallelism corresponding to distance E′F′ is E′F′T, which is complementary to angle DF′K. But angle DF′K is equal to angle ABC which is μ. Therefore EF is m', being identical with E′F′. See Fig. 3.

COROLLARY 2, THEOREM 1.

Given a length l to construct the corresponding angle of parallelism λ. (Bolyai's classical construction.[1])

Take a length AD equal to l. Draw DF at right angles to AD and of any length. Draw FE at right angles to DF and draw AE perpendicular from A on FE. Thus DF and AE are obtained. Construct a right-angled-triangle with DF as base and AE as hypotenuse. The angle opposite to the base is the required angle λ, as is obvious from Theorem 1.

THEOREM 2.

ABCD is a three-right-angled quadrilateral, having right angles at A, B and C. Along BC and BA take lengths BA′ and BC′ complementary to BA and BC, respectively, so that A′U the parallel to AD through A′ makes a right angle with BA′ and C′V the parallel to CD through C′ makes a right angle with BC′. Let EF be the common perpendicular to A′U and C′V, meeting C′V at E and A′U at F. See Fig. 4.

Then AD is equal to C′E and CD is equal to A′F.

[1] For another geometrical proof of Bolyai's Parallel Construction, see Liebmann, Nichteuklidische Geometrie, 2nd Edition, p. 35, or, Carslaw, Non-Euclidean Geometry, p 73.

PROOF.

To avoid complicacy of constructions it would be convenient to divide the proof into several distinct parts.

(1)

Let G be the middle point of CA′ and H of EF.

Then GH is parallel to CD. See Fig. 4.

Proof similar to that of corresponding part of Theorem 1.

(2)

From C′V and A′U cut off C′E′ and A′F′ equal to AD and CD, respectively. Let H′ be middle point of E′F′.

Then angle C′E′F′ is equal to angle A′F′E′. See Fig. 5 .

PROOF.

Consider the triangle E′F′D. Produce AD to X and CD to Y. The perpendicular bisectors of AC′ and A′C are also perpendicular bisectors of the sides F′D and E′D and are perpendicular to the common parallel p of AX and CY. Therefore the perpendicular to E′F′ at H′ is also porpendicular to p. Consequently from symmetry the angles C′E′F′ and A′F′E′ are equal.

(3)

GH′ is parallel to CD. See Fig. 6.

PROOF.

Produce BA′ to K making A′K equal to BC so that G the middle point of CA′ is also the middle point of BK. Draw KL at right angles to BK and make KL equal to BA. Join F′L. The quadrilaterals ABCD and LKA′F′ are congruent and the angles ADC and LF′A′ are equal. Produce LF′ to W. Then LW is parallel to CY as AX is parallel to A′U.

Join C′L. Let O be the middle point of C′L. Through O draw OZ parallel to C′V. See Fig. 7.

Draw C′M, LN, E′P, F′Q perpendiculars to OZ. Let B′M′ and K′N′ be common perpendiculars between BC′ and OZ, and KL and OZ. Then C′M is equal to LN, as OC′ is equal to OL. Therefore angle MC′V equals angle NLW, and because C′E′ equals LF′ the figures MC′E′P and NLF′Q are congruent. Therefore E′P is equal to F′Q. Consequently OZ passes through H′, the middle point of E′F′.

Again MC′B and NLK are equal being supplements of the equal angles MC′V and NLW. Consequently MC′B′M′ and NLK′N′ are congruent. Therefore B′M′ is equal to K′N′. The two figures BB′M′G and KK′N′G are congruent. Consequently BG is equal to GK, that is, OZ passes through G, the middle point of BK.

(4)

It follows from (1), (2), (3), that EF must coincide with E′F′.

This is established exactly as in the corresponding part of Theorem 1.

Thus Theorem 2 is completely proved.

Corollary 1, Theorem 2.

If we write the five elements a, b, c, λ, μ of a right-angled triangle in the order λ, μ, a, c, b there exists a rectangular pentagon whose sides in order are l, m, a', c, b'.

Suppose ABCD is the three-right-angled quadrilateral corresponding to a right-angled triangle with elements a, b, c, λ, μ so that AB, BC, CD, DA are equal to m', a, l, c and angle ADC is equal to β.

Construct the rectangular pentagon A′BC′EF as in Theorem 2. Then FA′ A′B, BC′, C′E are equal to l, m, a', c. The fifth side EF corresponds to angle of parallelism EFW which is complementary to angle UFW. But UFW is equal to XDY, that is β, so that EF is equal to b'. See Fig 8.

Corollary 2, Theorem 2.

With each vertex of the rectangular pentagon as origin we can re-construct a three-right-angled pentagon and from this again a right-angled triangle. The sides of the rectangular pentagon may be written in order in five different ways;

$l,\ m,\ a',\ c,\ b'$	(1)
$m,\ a',\ c,\ b',\ l$	(2)
$a',\ c,\ b',\ l,\ m$	(3)
$c,\ b',\ l,\ m,\ a'$	(4)
$b',\ l,\ m,\ a',\ c$	(5)

By identifying each of the sets (2), (3), (4), (5) with the set (1) we have five sets of possible values of a, b, c, λ, μ including the given set, *viz.*,

$$a, \quad b, \quad c, \quad \lambda, \quad \mu$$

$$c', \quad l', \quad b', \quad \mu, \quad \frac{\pi}{2}-a$$

$$b, \quad m', \quad l \quad \frac{\pi}{2}-a, \gamma$$

$$l', \quad a, \quad m, \quad \gamma, \quad \frac{\pi}{2}-\beta$$

$$m', \quad c', \quad a' \quad \frac{\pi}{2}-\beta, \lambda$$

We have thus the closed series of 5 associated right-angled triangles and the Engel-Napier Rules are shewn to possess a real geometrical basis in the rectangular pentagon.

Torsional Vibrations of a Circular Tube

BY

J. Ghosh.

1. The problem of the vibrations of cylinders has been discussed at great length by Rayleigh.[1] A particular solution in the case of the torsional vibrations of a solid cylinder has also been obtained. It is proposed in this paper to find the frequency equation in a more general case, *viz.*, when the solid is bounded by two co-axial cylinders and also when the thickness of the shell is small enough to be regarded as an infinitesimal of the first order.

2. Taking the axis of the cylinder as the axis of z and $(r,\theta.z)$ as the cylindrical coordinates of a point, the displacements of any point may be denoted by u_r, u_θ, u_z, which are usually assumed to be of the forms

$$\left.\begin{aligned} u_r &= \mathrm{U}e^{i(\gamma z+pt)} \\ u_\theta &= \mathrm{V}e^{i(\gamma z+pt)} \\ u_z &= \mathrm{W}e^{i(\gamma z+pt)} \end{aligned}\right\} \qquad \ldots \quad (1)$$

where U, V, W are independent of z and t.

3. In our present problem, we have

$$u_r = u_z = 0 \text{ and } u_\theta = \mathrm{V}e^{i(\gamma z+pt)},$$

where V is a function of r only.

The equation of motion gives[2]

$$\frac{\partial^2 \mathrm{V}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathrm{V}}{\partial r} - \frac{1}{r^2}\mathrm{V} + k^2\mathrm{V} = 0 \qquad \ldots \quad (2)$$

[1] Theory of Sound, Vol. I, Chaps. VII, VIII.

[2] Love's Elasticity, Art. 200.

where

$$k^2 = \frac{p^2\beta}{\mu} - \gamma^2. \qquad \ldots \quad (3)$$

The solution of (2) is evidently

$$V = AJ_1(kr) + BY_1(kr) \qquad \ldots \quad (4)$$

where J_1 is the Bessel Function of the first kind and Y_1 the Bessel Function of the second kind, both of the first order.

The traction across any surface $r=r$ are given by

$$\widehat{rr} = \widehat{rz} = 0$$

and

$$\widehat{r\theta} = \mu\left[\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} \right]$$

Hence if the surface $r=r$ is free from tractions, we have

$$A\,\frac{\partial J_1(kr)}{\partial(kr)} + B\,\frac{\partial Y_1(kr)}{\partial(kr)} - \frac{1}{kr}\{AJ_1(kr) + BY_1(kr)\} = 0,$$

which, by means of identities

$$\frac{\partial J_n(z)}{\partial z} = J_{n-1}(z) - \frac{n}{z} J_n(z),$$

$$\frac{\partial Y_n(z)}{\partial z} = Y_{n-1}(z) - \frac{n}{z} Y_n(z),$$

reduces to

$$A\left[J_0(kr) - \frac{2}{kr}J_1(kr)\right] - B\left[Y_0(kr) - \frac{2}{kr}Y_1(kr)\right] = 0 \qquad \ldots \quad (5)$$

Writing the conditions at $r=a$ and $r=b$ and elinimating A and B from these conditions, we get

$$\frac{kaJ_0(ka) - 2J_1(ka)}{kaY_0(ka) - 2Y_1(ka)} = \frac{kbJ_0(kb) - 2J_1(kb)}{kbY_0(kb) - 2Y_1(kb)} \qquad \ldots \quad (6)$$

This equation determines k. We may get the value of γ from the conditions at the plane ends of the cylinder and the period p is then obtained by means of (3).

One solution of (6) is found to be $k=0$, and the corresponding solution for a cylinder clamped at $z=0$ and $z=l$, is

$$u_\theta=\left(C_s r+\frac{D_s}{r}\right)\sin\frac{n\pi z}{l}\cos\left(\frac{n\pi t}{l}\sqrt{\frac{\mu}{\rho}}+a\right)$$

For a solid cylinder, we must put $D=0$.

4. We consider two particular cases.

(i) *One of the boundaries rigidly fi ed.* If $r=b$ be a rigid boundary, we have from (4) and (5) the frequency equation

$$\frac{kaJ_0(ka)-2J_1(ka)}{kaY_0(ka)-2Y_1(ka)}=\frac{J_1(kb)}{Y_1(kb)}$$

(ii) *Thickness of the shell very small.* For a shell of radius a and thickness an infinitesimal of the first order, the frequency equation (6) may be replaced by the equation

$$\frac{\partial}{\partial a}\left[\frac{kaJ_0(ka)-2J_1(ka)}{kaY_0(ka)-2Y_1(ka)}\right]=0 \qquad \dots \quad (7)$$

or $$[kaY_0(ka)-2Y_1(ka)][J_0(ka)+kaJ'_0(ka)+2J'_1(ka)]$$

$$-[kaJ_0(ka)-2J_1(ka)][Y_0(ka)+kaY'_0(ka)-2Y'_1(ka)]=0.$$

By means of the identities

$$\frac{\partial J_0(x)}{\partial x}=-J_1(x),\quad \frac{\partial Y_0(x)}{\partial x}=-Y_1(x)$$

$$\frac{\partial J_1(x)}{\partial x}=J_0(x)-\frac{1}{x}J_1(x)$$

$$\frac{\partial Y_1(x)}{\partial x}=Y_0(x)-\frac{1}{x}Y_1(x)$$

the above equation reduces to

$$k^2a^2[J_0(ka)Y_1(ka)-J_1(ka)Y_0(ka)]=0.$$

Since the value of the expression within brackets is of the form $\frac{c}{ka}$, where c is independent of ka, we get $k=0$ and this is the only solution of the equation (7). It is noteworthy that the relation between p and γ and hence the value of p itself is, in the case of a thin shell, independent of the radius of the shell.

On the Figures of Equilibrium of two rotating Masses of fluid for the Exponential Potential $\frac{e^{-kr}}{r}$.

Part II.

BY

Abanibhusan Datta.

1. In the first part of this paper, published in the Bulletin of the Calcutta Mathematical Society Vol. IX, No. 2, pp. 59-70., January, 1919, I studied the figures of equilibrium of two rotating masses of fluid for the exponential potential $\frac{e^{-kr}}{r}$, and intended to give later on a detailed numerical calculations of the results obtained therein and some diagrams, illustrating those results.

The present paper comprises the said numerical examples and some diagrams, showing the sections of the figures of equilibrium.

2. In part 1, it has been shown that the equations of the two masses can be approximately written in the forms:

$$\frac{r}{a}=1+$$

$$\frac{\frac{20\pi}{k^2\sqrt{c}}\, i^{-\frac{1}{2}}\left(\mathrm{A}\cosh \mathrm{A}k-\frac{\sinh \mathrm{A}k}{k}\right)\mathrm{I}_{\frac{5}{2}}(ak)\,\mathrm{K}_{\frac{5}{2}}(lc)+\frac{1}{6}a^2\omega^2}{4\pi\left\{\frac{1}{k^2}\left(a\cosh ak-\frac{\sinh ak}{k}\right)\frac{e^{-ak}}{a}(ka+1)+af_{\frac{5}{2}}(ak)\,k_{\frac{5}{2}}(ak)\right.}\,p_2(\mu)$$

$$\frac{\frac{1}{12}a^2\omega^2 p_2^2(\mu)\cos 2\phi}{4\pi\left\{\frac{1}{k^2}\left(a\cosh ak-\frac{\sinh ak}{k}\right)\frac{e^{-ak}}{a}(ka+1)+af_{\frac{5}{2}}(ak)\,k_{\frac{5}{2}}(ak)\right.}$$

$$+\frac{\frac{1}{k^2}\left(\mathrm{A}\cosh \mathrm{A}k-\frac{\sinh \mathrm{A}k}{k}\right)\frac{2}{\sqrt{ca}}\,i^{\frac{21}{2}}\left(\frac{7}{2}\right)\mathrm{I}_{\frac{7}{2}}(ak)\,k_{\frac{7}{2}}(kc)}{\left(a\cosh ak-\frac{\sinh ak}{k}\right)\frac{e^{-ak}}{k^2a}(ka+1)+af_{\frac{7}{2}}(ak)\,k_{\frac{7}{2}}(ak)}+\ldots$$

From symmetry, the equation of other mass is

$$\frac{R}{A}=1+\frac{\frac{20\pi}{k^2\sqrt{c}}i^{-\frac{1}{2}}\left\{a\cosh ak-\frac{\sinh ak}{k}\right\}I_{\frac{5}{2}}(Ak)\,k_{\frac{5}{2}}(kc)+\frac{1}{6}A^2\omega^2P_2(\mu)}{4k\left\{\frac{1}{k^2}\left(A\cosh Ak-\frac{\sinh Ak}{k}\right)\frac{e^{-Ak}}{A}(ka+1)+Af_{\frac{5}{2}}(Ak)\,k_{\frac{5}{2}}(Ak)\right\}}$$

$$-\frac{\frac{1}{12}A^2\omega^2P_2^2(\mu)\cos 2\phi}{4\pi\left\{\frac{1}{k^2}\left(A\cosh Ak-\frac{\sinh Ak}{k}\right)\frac{e^{-Ak}}{A}(kA+1)+Af_{\frac{5}{2}}(Ak)\,k_{\frac{5}{2}}(Ak)\right\}}$$

$$+\frac{\frac{1}{k^2}\left(a\cosh ak-\frac{\sinh ak}{k}\right)\frac{2}{\sqrt{c}\,A}i^{\frac{11}{2}}\frac{7}{2}I_{\frac{7}{2}}(Ak)\,k_{\frac{7}{2}}(kc)}{\left(A\cosh Ak-\frac{\sinh Ak}{k}\right)\frac{e^{-Ak}}{k^2A}(kA+1)+Af_{\frac{7}{2}}(Ak)\,k_{\frac{7}{2}}(Ak)}$$

$$+\quad\ldots\quad\ldots$$

3. First let us consider the case when the masses are equal *i.e.* when $A=a$.

Let the distance between the centres be $c=2{\cdot}5a$. For these data, the figures of the two masses are similar in shapes.

When $ka={\cdot}5$, each is given by an equation of the form :

$$\left.\begin{aligned}\frac{r}{a}&=1+{\cdot}19706\,p_2(\mu)-{\cdot}021\,p_2^2(\mu)\cos 2\phi+{\cdot}01164\,p_3(\mu)\\ \frac{R}{A}&=1+{\cdot}19706\,P_2(\mu)-{\cdot}021\,P_2^2(\mu)\cos 2\phi+{\cdot}01164\,P_3(\mu)\end{aligned}\right\}\quad(1)$$

Again, when $A=a$, $c=2{\cdot}5a$, and $ka=1$ the figures are given by

$$\left.\begin{aligned}\frac{r}{a}&=1+{\cdot}16287\,p_2(\mu)-{\cdot}0144\,p_2^2(\mu)\cos 2\phi+{\cdot}02135\,p_3(\mu)\\ \frac{R}{A}&=1+{\cdot}16287\,P_2(\mu)-{\cdot}0144\,P_2^2(\mu)\cos 2\phi+{\cdot}02135\,P_3(\mu)\end{aligned}\right\}\quad(2)$$

Also, when $A=a$, $c=2{\cdot}5a$, and $ka=10$, the figures are given by

$$\left.\begin{aligned}\frac{r}{a}&=1+{\cdot}0002845\,p_2(\mu)-{\cdot}0000119\,p_2^2(\mu)\cos 2\phi+{\cdot}000014823\,p_3(\mu)\\ \frac{R}{A}&=1+{\cdot}0002845\,P_2(\mu)-{\cdot}0000119\,P_2^2(\mu)\cos 2\phi+{\cdot}000014823\,P_3(\mu)\end{aligned}\right\}\quad(3)$$

On plotting (1), [*cf.* Table I, and Fig. I], it is found that the figures are almost similar in shapes to those given by Darwin with similar data. If however, ka is increased, the figures tend to become more spherical in shapes as will appear from (1), (2) and (3). When ka is fairly large as in (3), the terms involving the different harmonics are almost of negligible order of smallness.

Similar peculiarities are also noticed when the masses are unequel. When $A=3a$, and $\frac{c}{a}=5{\cdot}3$ and $ka={\cdot}2$, it is found that the figures are given by

$$\left.\begin{aligned} \frac{r}{a}&=1+2881\, p_2(\mu)-{\cdot}0234\, p_2^2(\mu)\cos 2\phi-1{\cdot}079\, p_3(\mu) \\ \frac{R}{A}&=1+{\cdot}06558\, P_2(\mu)-0029\, P_2^2(\mu)\cos 2\phi-{\cdot}000272\, P_3(\mu) \end{aligned}\right\} \quad (4)$$

The figures, as given by (4) have been drawn [*cf.* Table II, Fig. II and III]. The curves, however, do not present any marked difference from those obtained by Darwin for Newton's law with similar numerical data.

TABLE I. (For Figure I).

θ	$\cdot 197\, p_2(\mu)$	$-\cdot 021\, p_2^2(\mu)$	$\cdot 01164\, p_3(\mu)$	r/a
0°	·197	0	·01164	1·209
15°	·1772	—·0042	·0094	1·1824
30°	·122	—·015	·0038	1·1108
45°	·049	—·0315	—·002	1·0155
60°	—·0246	—·0473	—·005	·941
75°	—·079	—·0485	—·004	·8685
90°	—·0985	—·063	0	·8385
105°	—·079	—·0485	·004	·9125
120°	—·0246	—·0473	·005	·951
135°	·049	—·0315	·002	1·0195
150°	·122	—·015	—·0038	1·103
165°	·1772	—·0042	—·0094	1·1636
180°	·197	—0	—·01164	1·185

TABLE II. (For Figures II and III).

θ	$\cdot 2881\, p_2$	$-\cdot 02348\, p_2^2$	$-\cdot 1079\, p_3$	r/a
0°	·2881	0	—·1079	1·18
15°	·2591	—·0047	—·0868	1·1676
30°	·18	—·0176	—·035	1·1274
45°	·072	—·035	·019	1·056
60°	—·036	—·0528	·047	·9582
75°	—·1151	—·0554	+·037	·8667
90°	—·14405	—·07	0	·786
105°	—·1151	—·0554	—·007	·8
120°	—·036	—·0528	—·047	·8642
135°	·072	—·035	—·019	1·018
150°	·18	—·0176	+·035	1·1974
165°	·2591	—·0047	·0868	1·3412
180°	·2881	0	·1079	1·396
...	$\cdot 06558\, P_2$	$-\cdot 0029\, P_2^2$	$\cdot 000272\, P_3$	R/A
0°	·06558	0	·000272	1·0658
15°	·05899	—·000058	·00022	1·059
30°	·0401	—·002175	·000088	1·038
45°	·0164	—·00435	—·00005	1·012
60°	—·0082	—·0065	—·00012	·9852
75°	—·0262	—·0081	—·00094	·965
90°	—·0328	—·0087	0	·9585
105°	—·0262	—·0081	·00094	·967
120°	—·0082	—·0065	·00012	1·0146
135°	·0164	—·00435	·00005	1·012
150°	·0401	—·002175	—000088	1·038
165°	·05899	—·00058	—00022	1·058
180°	·06558	...	—000272	1·0653

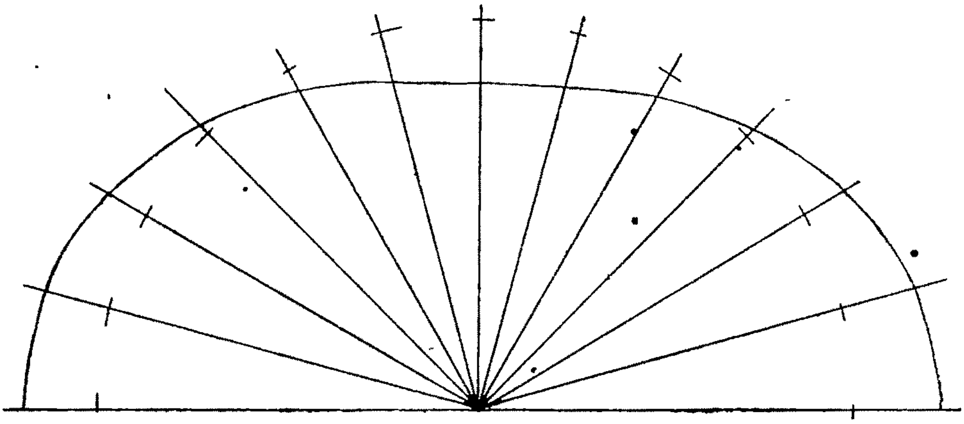


Fig. I

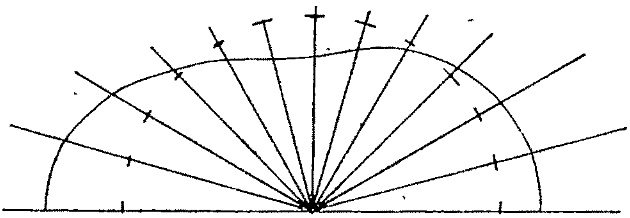


Fig. II

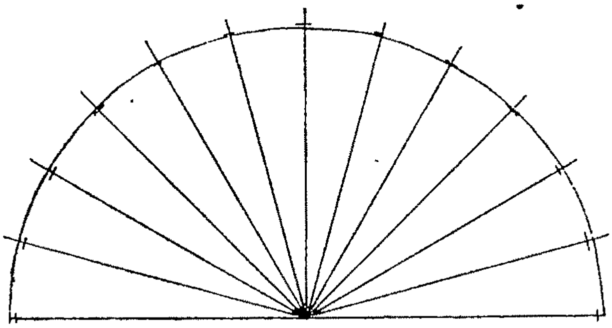


Fig. III

Notes and News

Professor C. E. Cullis, M.A., Ph.D., D.Sc., a founder Vice-President of the Calcutta Mathematical Society has retired from Indian Educational Service under the Government of Bengal and has left India for England. He is one of those few who felt early the necessity of a Mathematical Society in Calcutta and since its foundation always directed their best endeavours to the progress of the Society. He witnessed "the society grow up from an enthusiastic beginning into a sturdy adolescent challenging the attention of the world," and though he will be living and working in a distant land, we have every reason to hope that the welfare of the Society, for which he has laboured so earnestly and successfully, will always find a cherished corner in his heart.

While a member of the Indian Educational Service, he was offered and he readily accepted, the Hardinge Professorship of Higher Mathematics in the University of Calcutta. His acceptance of the Chair entailed an appreciable loss of income. But the appointment was most welcome to him as it afforded him more leisure to carry on his erudite researches into the Calculus of "Matrices and Determinoids." He held the Chair from 1917 to 1922 being reappointed in 1920. During these years, he delivered lectures according to the term of his appointment, on "the supremely condite subject," in which he was carrying on researches. These were all original and have been embodied in three volumes, two of which have been printed and published by the Cambridge University Press as also the first three chapters of the third volume. The University of Calcutta in conferring on him the honorary title of Doctor of Science has honoured herself. He has gone back to England not to rest on his oars having rendered long and eminent services to the cause of mathematical education in this country but to work uninterruptedly in the climate best suited to his health so that his monumental work on the Calculus of Matrices and Determinoids may be completed. We understand that he is now engaged on geometrical applications of the Calculus. Dr. Cullis considers that he will be able to complete his

work in 5 years and he intends to come back to India once more to deliver a new series of lectures on the applications of the Calculus he has developed.

Those who had the honour and pleasure of his acquaintance have the highest admiration for the man and the scholar. Dr. Cullis to them is the ideal of "plain living and high thinking."

We all earnestly pray to God Almighty to grant him a long lease of happy and prosperous life and we sincerely wish him ample leisure and strong health to enable him to complete his erudite and original researches which are so dear to him.

A meeting of the Society was fixed for Sunday, the 25th March, 1923 for the farewell of Prof. C. E. Cullis on the eve of his departure from India. It is most regrettable that he was prevented from attending the meeting of the Society by the sudden alteration of the date of his embarkation from the 26th to the 24th March.

The following resolution, moved by the Hon'ble Justice Sir Asutosh Mookerjee from the chair was adopted :—

That the Society place on record their high appreciation of the eminent services rendered by Prof. Cullis to the cause of mathematical learning in Calcutta, and to the Calcutta Mathematical Society and the Calcutta University in particular.

The following letter received from him on the same day, was read out in the meeting and was much appreciated by the members :—

4-2, Middleton Street,
Calcutta,
March 23rd, 1923.

To

The SECRETARY,

Calcutta Mathematical Society.

Dear Dr. Datta,

Will you please express to the members of the Calcutta Mathematical Society my intense regret that I shall be prevented from attending the meeting of the Society fixed for Sunday, March 25th

by the alteration of the date of my embarkation from the 26th to the 24th. Will you also convey to them my very grateful thanks for the tastefully and strongly bound volumes of the Bulletin which have been presented to me. No more acceptable present could possibly have been chosen; and the volumes will form one of the greatest treasures in my library at home.

Besides thanking my friends and fellow members of the Calcutta Mathematical Society for this present, I desire to leave with them my most sincere wishes for their welfare individually and for the welfare of the Society as a whole. I have watched the Society grow up from an enthusiastic beginning into a sturdy adolescent challenging the attention of the world. Since it will happily continue to receive the fostering care and stimulating encouragement of our honoured president, who called it into being, it has every prospect of a glorious maturity won by the earnest labours of its members. It is needless to say that no Mathematical literature will be more keenly watched by me than the succeeding numbers of the Bulletin, and the papers of my friends which I shall see in it. Through the Bulletin these friends will be almost as near to me as if I were in Calcutta.

Any man who can call himself a member of the Calcutta Mathematical Society will have a strong claim to a share in the gratitude I owe to the Society and to the University of Calcutta.

Yours very sincerely,

C. E. CULLIS,

(*Member of the Calcutta Mathematical Society*).

The All-India Mathematical Conference has been invited by the Calcutta Mathematical Society to hold its first session in Calcutta during Easter Holidays, 1923. But, that the time at the disposal of the Society being too short to raise sufficient funds and to arrange it satisfactorily, it was decided to postpone it to October, 1923. The members, we earnestly hope, will send in their contribution to our Treasurer, to the Conference Fund at their earliest convenience.

The following letter received by Prof. S. D. Mookerjee will, we hope, of some interest to our members and readers :—

INSTITUTE DE FRANCE 25, RUE HUMBOLDT,
February 23, 1923.

DEAR SIR,

I have duly received your little pack of memoirs though not containing the note on visual representation of 4-Dimensional Space alluded to in your letter, and which I think a certainly valuable one.[1]

The memoirs on osculating conics, or even non-euclidean geometry and parametric formulae in differential geometry of curves are less in my line, though I let you know, as likely to interest you, that a young Swiss Geometer, Mr. Juvet, generalized the Serret-Frenet formulae, starting from the relativity point of view, *i.e.,* from Riemannian Matrices.

But my interest in your New Methods in the Geometry of a Plane Arc, which I had expressed in 1909, in a (anonymous) note in the Revue Generale des Sciences has far from diminished since that time.

Precisely, at my Seminar or Colloquum, of the College de France, we have reviewed such subjects, and all my auditors and colleagues have been keenly interested by your way of researches, which we all consider as one of the most important roads to Mathematical Science.

The interest of these researches has been increased by comparison with memoires published in a slightly different line by a Dane, Mr. Juel, the list of which memoires I sent herewith, hoping that they may interest you. Indeed the conjunction of both kinds of works (Mr. Juel dealing with Straight Lines, you with circles and conics) is likely, at my sense to prove of great power and bearing for the further improvement of geometry.

With my congratulations for your beautiful results, I beg you to believe me

Yours truly,
J. HADAMARD.

Prof. Montel, who has reviewed at the College of France, Juel's memoirs, is going to deal with the subject in the Bulletin de Sciences Mathematiques, and intends to point out the importance of your notes.

[1] I have been mistaken in telling you that I had not got your paper on 4-Dimensional Space. They had been temporarily lost. I have them again and read with much interest.

Reviews

Introduction a la Théorie de la Relativite Calcul differential abéshe et Géométrie: Par Dr. H. Galbrun—The theory of generalised relativity is one of the wonders of recent times. 'Much admired, but little understood' is the language applied to it. Expositions of the theory from a variety of standpoints—philosophical, mathematical and physical—are streaming out from the pens of masterful writers. The theory, nevertheless, is likely to remain obscure to all as are not prepared, in the words of Lane, to elaborate their own understanding of it. It is of great help, it may be remarked, in comprehending a new theory, to study work that has preceded and in a sense prepared the way for it. Viewed from this standpoint, the work of Levi Civita and Ricci on what has now come to be known as differential geometry may be recommended to any student of relativity who is out to acquire more than a surface acquaintance with the subject. The work of these Italian writers, however, remains still mostly tucked away in Journals and it is mainly to the task of presenting them in a handy book—say, of reference—that Dr. H. Galbrun has addressed himself in his excellent volume entitled 'Introduction a la Théorie de la Relativité etc.' Riemann space of four dimensions—n-dimensions, rather, for the author prefers to use the n-dimensional notation for the sake of generality—is regarded as embedded in Euclidian space of higher dimensions and the properties of tensors, geodesics etc. are worked out on this basis. The neat summational notation used by Einstein is dropped in favour of the usual method and this is a little unfortunate as nothing is gained thereby in the way of relieving the reader of that mental work or pen-work which he, as the author rightly says, has to get through at every step before he is ready for the next. The author, it is right to add, has a chapter on Weyl's geometry including his concept of 'Eich-invarianz' and the modification thereof suggested by Prof. Eddington. The author then gives a discussion of the pre-relativity mechanics and electrodynamics of the Galilian spaces and winds up with a detailed examination of the restricted principle of Einstein-Minkowski based on Euclidian geometry of four-dimensions. (S. C. K.)

Calcutta Mathematical Society

REPORT FOR THE YEAR 1922.

The gentlemen named below were the office-bearers and members of the Council for the year 1922:

PRESIDENT.

The Hon'ble Justice Sir Asutosh Mookerjee, Saraswati, Kt., C.S.I., M.A., D.L., D.Sc., Ph.D., F.R.A.S., F.R.S.E.

VICE-PRESIDENTS.

The Hon'ble Mr. Mahendranath Ray, M.A., B.L., C.I.E.
Prof. C. E. Cullis, M.A., Ph.D., D.Sc.
Prof. Syamadas Mookerjee, M.A., Ph.D.

TREASURER.

Rai A. C. Bose, Bahadur, M.A.

SECRETARY.

S. K. Banerjee, Esq., D.Sc. (up to April, 1922).
Abanibhusan Datta, Esq., M.A., Ph.D. (from April, 1922).

COUNCIL.

D. N. Mallik, Esq., D.Sc., F.R.S.E.
Principal S. C. Bagchi, LL.D., Bar-at-law.
Prof. C. V. Raman, M.A., D.Sc.
Satishchandra Bose, Esq., M.A.
Manmathanath Ray, Esq., M.A., B.L.
Narendrakumar Mazumdar, Esq., M.A.
Phanindralal Ganguli, Esq., M.A., B.L.
Surendramohan Ganguli, Esq., D.Sc.
Bibhutibhusan Datta, Esq., D.Sc.

During the year 1922, four meettings were held, and 35 papers were read. Four issues of the Bulletin (*viz.,* Vol. XII, No. 4, Vol.

XIII, Nos. 1, 2, 3) containing 22 original papers, reviews, notices and miscellaneous notes have been published in 1922. The number of the Bulletin which was to have been published in June, 1922, could not be brought out in time and consequently a double number was issued in September. The four numbers issued during the year cover over 250 pages.

Dr. S. K. Banerji, the previous Secretary resigned his office in April, 1922, as he joined the Meteorological Dept. of the Government of India.

The Society has entered into several exchange relations during the year and has also received during the year some new orders to supply the Bulletin from several distinguished Universities and Libraries in different parts of the world.

The Calcutta Mathematical Society invited the All-India Mathematical Conference to hold its first session in Calcutta during the next Easter Holidays. This has since been postponed to October, 1923.

The Society takes this opportunity of again placing on record its heartfelt gratitude to its distinguished President for his fostering care and unflagging zeal for the advancement of the Society. A true friend and champion of the cause of higher learning and an erudite mathematician, the Society receives from him unfailing encouragement and advice in the dissemination of mathematical knowledge. The Society tenders its best thanks to its former energetic Secretary for his signal services and to other office-bearers for the ungrudging interest taken by them for the welfare of the Society.

The best thanks of the Society are also due to the Superintendent and staff of the Calcutta University Press for the care and promptitude with which they printed the Bulletin.

Seven ordinary members were elected in 1922.

The following papers were read before the meetings during the year under review :—

1. Prof. G. H. Bryan, Sc.D., F.R.S. : "Note on the graphical solution of Spherical Triangles."
2. Prof. C. V. Raman : "Quantum Theory of Light."
3. Prof. S. K. Banerji : "On the depth of Earthquake focus."
4. Prof. S. K. Banerji : "On the solution of the equation $\nabla^4\psi=0$ in bipolar co-ordinates."
5. Prof. C. V. Hanumanto Rao : "Fundamental relations in homogeneous co-ordinates."
6. Prof. C. V. Hanumanto Rao : "On the ϕ conic of two conics."

7. Mr. Bidhubhusan Ray : "The optical analogue of the whispering gallery effect."
8. Mr. G. Bhar : "The oscilating conic at infinity."
9. Pandit Oudh Upadhyaya : "On an Algebraical Identity."
10. Pandit Oudh Upadhyaya : "On the values of the polynomials in a transformation formula for $\frac{x^p-1}{x-1}$ where p is any prime number of the form $2n+1$."
11. Mr. Nalinikanta Basu : "The stability of a dirigible baloon."
12. Mr. Nripendranath Sen : "On some problems in tidal oscillations."
13. Mr. B. N. Chuckerbutti : "The non-radiating electronic orbits and the normal Zeeman effect."
14. Mr. P. Das : "Caustics formed by diffraction."
15. Prof. C. V. Raman : "Boltzmann's principle and some of its applications."
16. Dr. Philip Franklin, New York, U. S. A. : "On the generalised angle Concept."
17. Mr. Nripendranath Sen : "On a steady motion of a viscous fluid due to the rotation of two spheroids about their common axis."
18. Mr. Nripendranath Sen : "On the motion of two spheroids in infinite liquid."
19. Mr. N. M. Basu : "On some laws of Central Force."
20. Mr. P. Das : "Note on Rydberg's constant."
21. Mr. P. Das : "On the secondary spectrum of Hydrogen."
22. Mr. N. K. Basu : "On the steering of an aeroplane in a horizontal circle.
23. Mr. G. Bhar : "The osculating conic in homogeneous coordinates."
24. Mr. K. Basu : "Note on certain properties of Legendre polynomials of the second type."
25. Mr. K. Basu : "On the product of Bessel Functions."
26. Mr. K. Basu : "On the steady motion of a viscous fluid in a semi-infinite space bounded by a plane due to a rotating circular cylinder in front of it."
27. Dr. A. B. Datta : "On an application of Bessel Functions to probability."
28. Mr. Satishchandra Chakrabarty : "On the evaluation of some factorable continuants."

29. M. Maurice Frechet: "Outlines of a theory of abstract aggregates."
30. Mr. M. Ghosh: "Expressions for the radius of a circle in areal co-ordinates."
31. Mr. N. N. Sen: "On Vertex rings of finite section."
32. Mr. N. N. Ghosh: "Algebra of Polynomials."
33. Mr. S. C. Mitra: "On waves due to a submerged elliptic cylinder."
34. Mr. S. C. Mitra: "Liquid motion inside rotating arcs of three and four circles."
35. Mr. K. Basu: "On magnetic field due to a Thermionic Valve."

APPENDIX A.

Income and Expenditure Account, 1922.

Income.	Rs.	a.	p.	Expenditure.	Rs.	a.	p.
Opening Balance ...	586	11	0	Books and Periodicals ...	29	12	0
Admission Fees ...	20	0	0	Binding Charges ...	nil		
Subscriptions ...	723	0	0	Furniture	6	8	0
Sale Proceeds of Bulletin, etc.	198	4	0	Stationery	1	4	0
				Printing, including cover papers for Bulletin ...	83	0	0
				Despatching charges ...	15	5	6
				General charges, including Secretary's expenses ...	90	13	6
				Meeting expenses ...	21	4	0
				Establishment (December 1921 to December 1922 inclusive)	638	0	0
				Closing Balance with Secretary	10	14	9
				Closing Balance with Treasurer	631	1	3
TOTAL ...	1,527	15	0	TOTAL ...	1,527	15	0